Chapitre 22

TD

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Exercice 3

$$f : (x,y) \mapsto \begin{cases} \dfrac{xy}{x^2+y^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{sinon .} \end{cases}$$

Si $f(x,y) \xrightarrow[(x,y)\to(0,0)]{} 0$, alors si $x, y : I \to \mathbb{R}$ telles que $\begin{cases} x(t) \xrightarrow[t\to 0]{} 0 \\ y(t) \xrightarrow[t\to 0]{} 0 \end{cases}$ alors $f\big(x(t), y(t)\big) \xrightarrow[t\to 0]{} 0$.

On a $x(t) = t$ et $y(t) = t$. On calcule donc

$$f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2} \xrightarrow[x\to 0]{} \frac{1}{2} \neq 0.$$

Donc

$$f(x,y) \underset{(x,y)\to(0,0)}{\not\longrightarrow} (0,0)$$

et donc f n'est pas continue en $(0,0)$.

Exercice 2

$$f(x,y) = \frac{3x^2 + xy}{\sqrt{x^2+y^2}}.$$

Soit $\varepsilon > 0$. On cherche $r > 0$ tel que

$$\underbrace{\forall (x,y) \in B_{(0,0)}(r) \setminus \{(0,0)\}}_{0<\sqrt{x^2+y^2}<r}, |f(x,y) - 0| \leqslant \varepsilon.$$

Soit $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\begin{aligned} \left|3x^2 + xy\right| &\leqslant 3x^2 + |xy| \\ &\leqslant 3(x^2+y^2) + \frac{1}{2}(x^2+y^2) \\ &\leqslant \frac{7}{2}\|(x,y)\|^2 \end{aligned}$$

donc

$$\left|\frac{3x^2+xy}{\sqrt{x^2+y^2}}\right| \leqslant \frac{7}{2}\|(x,y)\|.$$

On pose $r = \frac{2}{7}\varepsilon > 0$. Soit $(x,y) \in B_{(0,0)}(r) \setminus \{(0,0)\}$.

Alors

$$|f(x,y)| \leqslant \frac{7}{2}\|(x,y)\| < \frac{7}{2}r = \varepsilon.$$

Donc

$$f(x,y) \xrightarrow[(x,y)\to(0,0)]{} 0.$$

Exercice 6

1. On pose $g : (x,y) \mapsto f(y,x)$.

$$\begin{aligned} \frac{\partial g}{\partial x}(x,y) &= \frac{\partial f}{\partial x}(y,x)\,\frac{\partial y}{\partial x}(x,y) + \frac{\partial f}{\partial y}(y,x)\,\frac{\partial x}{\partial x}(x,y) \\ &= \frac{\partial f}{\partial y}(y,x) \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial y}(x,y) &= \frac{\partial f}{\partial x}(y,x)\,\frac{\partial x}{\partial x}(x,y) + \frac{\partial f}{\partial y}(y,x)\,\frac{\partial y}{\partial x}(x,y) \\ &= \frac{\partial f}{\partial x}(y,x) \end{aligned}$$

2. On pose $h : (x,y) \mapsto f\big(y, f(x,x)\big)$ et $k : x \mapsto f(x,x)$

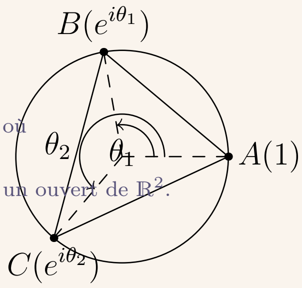
$$\begin{aligned}\frac{\partial h}{\partial x}(x,y) &= \frac{\partial f}{\partial x}\big(y, f(x,x)\big)\ \frac{\partial y}{\partial x}(x,y) + \frac{\partial f}{\partial y}\big(y, f(x,x)\big)k'(x)\\ &= \frac{\partial f}{\partial y}\big(y, f(x,x)\big)\left(\frac{\partial f}{\partial x}(x,x) + \frac{\partial f}{\partial y}(x,x)\right)\end{aligned}$$

En effet, on a $\gamma : x \mapsto (x,x)$ et $\gamma'(x) = (1,1)$.
Donc $f \circ \gamma : x \mapsto f(x,x) = k(x)$.

$$\begin{aligned}k'(x) &= \big\langle \nabla f(x,x) \mid \gamma'(x)\big\rangle\\ &= \frac{\partial f}{\partial x}(x,x) + \frac{\partial f}{\partial y}(x,x).\end{aligned}$$

$$\begin{aligned}\frac{\partial h}{\partial y}(x,y) &= \frac{\partial f}{\partial x}\big(y, f(x,x)\big)\frac{\partial f}{\partial y}(x,y) + \frac{\partial f}{\partial y}\big(y, f(x,x)\big) \times 0\\ &= \frac{\partial f}{\partial x}\big(y, f(x,x)\big).\end{aligned}$$

Exercice 10



$$\begin{aligned} f : U &\longrightarrow \mathbb{R}\\ (\theta_1,\theta_2) &\longmapsto \left|1-e^{i\theta_1}\right| + \left|e^{i\theta_1}-e^{i\theta_2}\right| + \left|1-e^{i\theta_2}\right|\end{aligned}$$

où $U = \{(\theta_1,\theta_2) \in]0,2\pi[^2 \mid \theta_1 < \theta_2\}$

un ouvert de $\mathbb{R}^2$.

On a

$$\begin{cases}\left|1-e^{i\theta_1}\right| = 2\left|\sin\dfrac{\theta_1}{2}\right|,\\ \left|1-e^{i\theta_2}\right| = 2\left|\sin\dfrac{\theta_2}{2}\right|,\\ \left|e^{i\theta_1}-e^{i\theta_2}\right| = 2\left|\sin\dfrac{\theta_1-\theta_2}{2}\right|.\end{cases}$$

Soit $(\theta_1,\theta_2) \in U$. Donc,

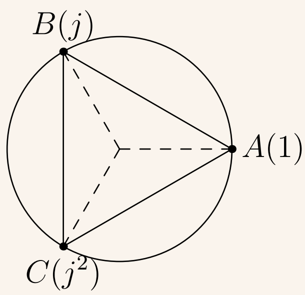
$$f(\theta_1,\theta_2) = 2\left(\sin\left(\frac{\theta_1}{2}\right) + \sin\left(\frac{\theta_2}{2}\right) + \sin\left(\frac{\theta_2-\theta_1}{2}\right)\right)$$

On a

$$\begin{cases}\dfrac{\partial f}{\partial \theta_1}(\theta_1,\theta_2) = 2\left(\dfrac{1}{2}\cos\left(\dfrac{\theta_1}{2}\right) - \dfrac{1}{2}\cos\left(\dfrac{\theta_2-\theta_1}{2}\right)\right),\\ \dfrac{\partial f}{\partial \theta_2}(\theta_1,\theta_2) = 2\left(\dfrac{1}{2}\cos\left(\dfrac{\theta_2}{2}\right) + \dfrac{1}{2}\cos\left(\dfrac{\theta_2-\theta_1}{2}\right)\right).\end{cases}$$

On cherche (θ_1, θ_2) tel que

$$\begin{cases} \cos\frac{\theta_1}{2} - \cos\frac{\theta_2-\theta_1}{2} = 0 \\ \cos\frac{\theta_2}{2} + \cos\frac{\theta_2-\theta_1}{2} = 0 \end{cases}$$
$$\iff \begin{cases} \frac{\theta_2}{2} = \frac{\theta_2-\theta_1}{2} \\ \pi - \frac{\theta_2}{2} = \frac{\theta_2-\theta_1}{2} \end{cases}$$
$$\iff \begin{cases} 2\theta_1 = \theta_2 \\ 2\theta_2 - \theta_1 = 2\pi \end{cases}$$
$$\iff \begin{cases} \theta_1 = \frac{2\pi}{3} \\ \theta_2 = \frac{4\pi}{3} \end{cases}$$



Exercice 4

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \begin{cases} 2x^2+y^2-1 & \text{si } x^2+y^2>1 \\ x^2 & \text{sinon} \end{cases} \end{aligned}$$

On regarde la continuité sur chacun des morceaux ouverts, puis sur les points du bord.

Soit $(x_0, y_0) \in \mathbb{R}^2$.

Cas 1 On suppose que ${x_0}^2 + {y_0}^2 > 1$.
On doit montrer :

$$\boxed{\forall \varepsilon > 0, \exists r > 0, \forall (x,y) \in \mathbb{R}^2, \quad \|(x,y)-(x_0,y_0)\| \leqslant r \implies |f(x,y) - f(x_0,y_0)| \leqslant \varepsilon}$$

Soit (x,y) vérifiant $x^2 + y^2 > 1$.

$$\begin{aligned} |f(x,y) - f(x_0,y_0)| &= \left|2x^2 + y^2 - 1 - 2x_0^2 - y_0^2 + 1\right| \\ &= \left|2(x^2 - x_0^2) + (y^2 - y_0^2)\right| \\ &\leqslant 2|x^2 - x_0^2| + |y^2 - y_0^2| \\ &\leqslant 2\|(x,y)-(x_0,y_0)\| + \|(x,y)-(x_0,y_0)\| \quad \leqslant 3\|(x,y)-(x_0,y_0)\| \end{aligned}$$

On pose $R = \sqrt{x_0^2 + y_0^2} - 1$. Soit $\varepsilon > 0$. On pose $r = \dfrac{1}{2}\min\left(\dfrac{\varepsilon}{3}, R\right)$.
Ainsi, $r < R$ et $r < \dfrac{\varepsilon}{3}$.
Soit $(x, y) \in B_{(x_0,y_0)}(r)$. Alors, comme $r < R$, $x^2 + y^2 > 1$ et donc

$$\left|f(x, y) - f(x_0, y_0)\right| \leqslant 3r < \varepsilon.$$

<u>Cas 2</u> $x_0^2 + y_0^2 < 1$.
<u>Cas 3</u> On suppose que $x_0^2 + y_0^2 = 1$. Soit $(x, y) \in \mathbb{R}^2$.
Si (x, y) vérifie $x^2 + y^2 > 1$:

$$\begin{aligned}\left|f(x, y) - f(x_0, y_0)\right| &= \left|2x^2 + y^2 - 1 - x_0^2\right| \\ &\leqslant \left|2x^2 + y^2 - x_0^2 - y_0^2 - x_0^2\right| \\ &\leqslant \left|2(x^2 - x_0^2) + y^2 - y_0^2\right| \\ &\leqslant 3\|(x, y) - (x_0, y_0)\|.\end{aligned}$$

Si $x^2 + y^2 \leqslant 1$:

$$\left|f(x, y) - f(x_0, y_0)\right| = \left|x^2 - x_0^2\right| \leqslant \|(x, y) - (x_0, y_0)\|$$

Soit $\varepsilon > 0$. On pose $r = \dfrac{\varepsilon}{6} < \dfrac{\varepsilon}{3}$ et donc

$$\forall (x, y) \in B_{(x_0,y_0)}(r), \left|f(x, y) - f(x_0, y_0)\right| \leqslant \varepsilon.$$

Exercice 7

Soit $u = (a, b) \in \mathbb{R}^2$.

$$\forall t \neq 0, \Delta f(t) = \frac{f\big((0, 0) - tu\big) - f(0, 0)}{t} = \frac{1}{t} f(ta, tb).$$

<u>Cas 1</u> $a \neq 0$. Alors,

$$\forall t \neq 0, \Delta f(t) = \frac{1}{t} t^2 b^2 \ln\left(|ta|\right) = tb^2 \ln\left(|ta|\right)\big) \xrightarrow[t \to 0]{} 0.$$

<u>Cas 2</u> $a = 0$. Alors,

$$\Delta f(t) = \frac{1}{t} \times 0 = 0 \xrightarrow[t \to 0]{} 0.$$

On trouve que

$$\forall u \in \mathbb{R}^2, \mathrm{d}f(u)(0, 0) = 0.$$

On pose

$$\begin{cases} x(t) = t \\ y(t) = \begin{cases} 0 & \text{si } t = 0 \\ \sqrt{\dfrac{1}{\left|\ln\left(|t|\right)\right|}} & \text{si } t \neq 0 \end{cases} \end{cases}$$

et on a bien

$$\begin{cases} x(t) \xrightarrow[t \to 0]{} 0 \\ y(t) \xrightarrow[t \to 0]{} 0 \end{cases}$$

On a

$$\lim_{t \to 0} f\big(x(t), y(t)\big) = -1 \xrightarrow[t \to 0]{} -1 \neq f(0, 0)$$

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On a

$$\begin{cases} \dfrac{\partial f}{\partial x}(0,0) = \mathrm{d}f(1,0)(0,0) = 0, \\ \dfrac{\partial f}{\partial y} = \mathrm{d}f(0,1)(0,0) = 0. \end{cases}$$

et

$$\forall (x,y) \in \mathbb{R}^* \times \mathbb{R}, \begin{cases} \dfrac{\partial f}{\partial x}(x,y) = \dfrac{y^2}{|x|} \times \operatorname{sgn}(x) \xrightarrow[x \to 0]{} \pm\infty \\ \dfrac{\partial f}{\partial y}(x,y) = 2y \ln\big(|x|\big). \end{cases}$$

Exercice 8

On a

$$\begin{cases} \dfrac{\partial g}{\partial x}(x,y) = \dfrac{1}{y}\left(e^{-x} - xe^{-x}\right) = \dfrac{e^{-x}}{y}(1-x) \\ \dfrac{\partial g}{\partial y}(x,y) = \dfrac{-xe^{-x}}{y^2} + \dfrac{1}{e} \end{cases}$$

$$\left.\begin{aligned} \frac{\partial g}{\partial x}(x,y) = 0 \\ \frac{\partial g}{\partial y}(x,y) = 0 \end{aligned}\right\} \iff \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$$\iff \begin{cases} x = 1 \\ y = 1 \end{cases}$$

On a

$$g(1,1) = \frac{1}{e} + \frac{1}{e} = \frac{2}{e}.$$

Or,

$$g(1,2) = \frac{1}{2e} + \frac{2}{e} > \frac{2}{e}$$

et

$$g\left(\frac{1}{2}, 1\right) = \frac{1}{2}e^{-\frac{1}{2}} + \frac{1}{e} < \frac{2}{e}$$

car $\dfrac{1}{2}e^{-\frac{1}{2}} < e^{-1} \iff \dfrac{1}{2} < e^{-\frac{1}{2}} \iff 2 > e^{\frac{1}{2}} \iff 4 > e.$

Donc, $(1,1)$ n'est ni un minimum, ni un maximum pour g.

Exercice 9

1. f est $\mathscr{C}^2$ donc
$$\forall h \in \mathbb{R}, f(h) = f(0) + hf'(0) + \underset{h\to 0}{o}(h)$$
donc
$$\forall (x,y) \in \mathbb{R}^2, f(x^2+y^2) = f(0) + (x^2+y^2)\, f'(0) + \underset{(x,y)\to(0,0)}{o}(x^2+y^2)$$
donc
$$\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, F(x,y) = f'(0) + \underset{(x,y)\to(0,0)}{o} \xrightarrow[(x,y)\to(0,0)]{} f'(0).$$
On pose donc $F(0,0) = f'(0)$.
2. f est de classe $\mathscr{C}^2$ donc
$$\forall (x,y) \in \mathbb{R}^2, f(x^2+y^2) = f(0)+(x^2+y^2)f'(0)+\frac{(x^2+y^2)^2}{2}f''(0)+\underset{(x,y)\to(0,0)}{o}\left((x^2+y^2)^2\right).$$
Donc
$$\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, F(x,y) = f'(0) + \frac{1}{2}(x^2+y^2)f''(0) + \underset{(x,y)\to(0,0)}{o}\left(x^2+y^2\right).$$
On en déduit que
$$\frac{\partial F}{\partial x}(0,0) = \lim_{\substack{x\to 0\\ \neq}} \frac{F(x,0)-F(0,0)}{x} = \lim_{\substack{x\to 0\\ \neq}} \left(\frac{1}{2}xf''(0) + \underset{x\to 0}{o}(x)\right) = 0$$
et
$$\frac{\partial F}{\partial y}(0,0) = \lim_{\substack{y\to 0\\ \neq}} \frac{F(0,y)-F(0,0)}{y} = 0.$$
Soit $(x,y) \neq (0,0)$.
$$\frac{\partial F}{\partial x}(x,y) = \frac{2xf'(x^2+y^2)(x^2+y^2) - 2x\big(f(x^2+y^2)-f(0)\big)}{(x^2+y^2)^2}$$
donc $\dfrac{\partial F}{\partial x}$ est continue sur $\mathbb{R}^2 \setminus \{(0,0)\}$. Or, f' est $\mathscr{C}^1$ donc
$$f'(x^2+y^2) = f'(0) + (x^2+y^2)f''(0) + o(x^2+y^2)$$
d'où
$$\begin{aligned}\frac{\partial F}{\partial x}(x,y) &= \frac{2x\big(f'(0)+(x^2+y^2)f''(0)+o(x^2+y^2)\big)}{x^2+y^2} - \frac{2x}{x^2+y^2}\left(f'(0) + \frac{x^2+y^2}{2}f''(0) + o(x^2+y^2)\right) \\ &= x\frac{f''(0)}{2} + o(1) \xrightarrow[(x,y)\to(0,1)]{} 0 = \frac{\partial F}{\partial x}(0,0)\end{aligned}$$
donc $\dfrac{\partial F}{\partial x}$ est continue en $(0,0)$. De même, $\dfrac{\partial F}{\partial y}$ est continue sur $\mathbb{R}^2$.
On en déduit que F est $\mathscr{C}^1$.

Exercice 11

Il existe 4 dérivées secondes :

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

On pose

$$\begin{aligned} g : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (u,v) &\longmapsto f\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right). \end{aligned}$$

Ainsi, $\forall (x,y) \in \mathbb{R}^2, g(x+y, x-y) = f(x,y)$.

Donc,

$$\begin{aligned} \frac{\partial g}{\partial u}(u,v) &= \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial u} \\ &= \frac{\partial f}{\partial x} \times \frac{1}{2} + \frac{\partial f}{\partial y} \times \frac{1}{2} \end{aligned}$$

On en déduit que

$$\begin{aligned} \frac{\partial^2 g}{\partial v\,\partial u} &= \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} \times \frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y\,\partial x} \times \frac{\partial y}{\partial v} + \frac{\partial^2 f}{\partial x\,\partial y}\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial v}\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\cancel{\frac{\partial^2 f}{\partial x^2}} - \frac{1}{2}\cancel{\frac{\partial^2 f}{\partial y\,\partial x}} + \frac{1}{2}\cancel{\frac{\partial^2 f}{\partial x\,\partial y}} - \frac{1}{2}\cancel{\frac{\partial^2 f}{\partial y^2}}\right) \end{aligned}$$

> **Théorème** (Schwarz)**:** Si f est $\mathscr{C}^2$ alors $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$.
>
> (vu l'année prochaine)

Donc $\dfrac{\partial^2 g}{\partial u\,\partial v} = 0$.

$$\frac{\partial g}{\partial v} = F(v)$$
$$g(u,v) = G(v) + H(u)$$
$$f(x,y) = G(x-y) + H(x+y)$$

Exercice 12

1. $\begin{cases} \dfrac{\partial f}{\partial x} = xy^2 \\ \dfrac{\partial f}{\partial y} = x^2 y \end{cases}$

Donc $f(x,y) = \dfrac{x^2y^2}{2} + \lambda$.

2. $\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}} \end{cases}$

Donc $f(x,y) = \sqrt{x^2+y^2} + \lambda$.

3. $\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{x}{x^2+y^2} \\ \dfrac{\partial f}{\partial y} = -\dfrac{y}{x^2+y^2} \end{cases}$

D'après (E_1), $f(x,y) = \dfrac{1}{2}\ln\left(x^2+y^2\right) + g(y)$

Or,

$$\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2} + g'(y) = -\frac{y}{x^2+y^2} \iff g'(y) = \frac{2y}{x^2+y^2} \text{ dépend de } x \ \lightning$$