

TD n° 10

*Preuves en logique
propositionnelle*

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1 Premiers arbres de preuves

1.

$$\frac{\text{Ax}}{p \vdash p} \rightarrow\text{i}.$$

2.

$$\frac{\frac{\text{Ax}}{p, \neg p \vdash p} \quad \frac{\text{Ax}}{p, \neg p \vdash \neg p}}{p, \neg p \vdash \perp} \neg\text{e}.$$

3.

$$\frac{\frac{\text{Ax}}{p, q \vdash p} \quad \frac{\text{Ax}}{p, q \vdash q}}{p, q \vdash p \wedge q} \wedge\text{i}.$$

4.

$$\frac{\frac{\frac{\text{Ax}}{p \wedge q \vdash p \wedge q} \wedge\text{e,g} \quad \frac{\text{Ax}}{p \wedge q \vdash p \wedge q} \wedge\text{e,d}}{p \wedge q \vdash q} \wedge\text{i} \quad \frac{\text{Ax}}{p \wedge q \vdash q \wedge p} \wedge\text{i}.$$

5.

$$\frac{\frac{\text{Ax}}{p \vee q \vdash p \vee q} \quad \frac{\frac{\text{Ax}}{p \vee q, p \vdash p} \vee\text{i,d} \quad \frac{\text{Ax}}{p \vee q, q \vdash q} \vee\text{i,g}}{p \vee q, p \vdash q \vee p} \vee\text{e}}{p \vee q \vdash q \vee p} \vee\text{e}.$$

6.

$$\frac{\frac{\frac{\text{Ax}}{p \wedge \neg p \vdash p \wedge \neg p} \wedge\text{e,g} \quad \frac{\text{Ax}}{p \wedge \neg p \vdash p \wedge \neg p} \wedge\text{e,d}}{p \wedge \neg p \vdash \perp} \neg\text{e} \quad \frac{\text{Ax}}{p \wedge q \vdash \perp} \neg\text{i}}{\vdash \neg(p \wedge \neg p)} \neg\text{i}.$$

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2 Divers arbres de preuves

1.

$$\frac{\frac{\text{Ax}}{p \vee (p \wedge q) \vdash p \vee (p \wedge q)} \quad \frac{\text{Ax}}{p \vee (p \wedge q), p \vdash p} \quad \frac{\frac{\text{Ax}}{p \vee (p \wedge q), p \vee q \vdash p \wedge q} \wedge\text{e,g}}{p \vee (p \wedge q), p \vee q \vdash p} \vee\text{e}}{p \vee (p \wedge q) \vdash p} \vee\text{e}.$$

2.

$$\frac{\frac{\frac{\text{Ax}}{p \wedge q, r \wedge s \vdash p \wedge q} \wedge\text{e,g} \quad \frac{\text{Ax}}{p \wedge q, r \wedge s \vdash r \wedge s} \wedge\text{e,d}}{p \wedge q, r \wedge s \vdash p} \wedge\text{i} \quad \frac{\text{Ax}}{p \wedge q, r \wedge s \vdash r} \wedge\text{i}}{p \wedge q, r \wedge s \vdash p \wedge r} \wedge\text{i}.$$

3.

$$\frac{\frac{\frac{\text{Ax}}{p, q \wedge r \vdash p} \quad \frac{\frac{\text{Ax}}{p, q \wedge r \vdash q \wedge r} \wedge\text{e,g}}{p, q \wedge r \vdash q} \wedge\text{i}}{p, q \wedge r \vdash p \wedge q} \wedge\text{i} \quad \frac{\text{Ax}}{p, q \wedge r \vdash p \wedge q} \wedge\text{i}.$$

4.

$$\frac{\frac{\frac{\text{Ax}}{p, \neg p \vdash p} \quad \frac{\text{Ax}}{p, \neg p \vdash \neg p} \neg\text{e}}{p, \neg p \vdash \perp} \neg\text{i}}{p \vdash \neg\neg p} \neg\text{i}.$$

5.

$$\frac{\frac{\frac{\text{Ax}}{p, \neg\neg p, \neg p \vdash p} \quad \frac{\text{Ax}}{p, \neg\neg p, \neg p \vdash \neg p} \neg\text{e}}{p, \neg\neg p, \neg p \vdash \perp} \neg\text{i} \quad \frac{\text{Ax}}{p, \neg\neg p, \neg p \vdash \neg\neg p} \neg\text{e}}{\frac{\text{Ax}}{p, \neg\neg p \vdash \perp} \neg\text{i}} \neg\text{i}.$$

3 Lois de DE MORGAN

1.

$$\frac{\frac{\frac{\text{Ax}}{\neg(p \vee q), p \vdash p} \quad \text{vi,g}}{\neg(p \vee q), p \vdash p \vee q} \quad \frac{\frac{\text{Ax}}{\neg(p \vee q), p \vdash \neg(p \vee q)} \quad \text{Ax}}{\neg(p \vee q), p \vdash \perp} \quad \neg\text{i}}{\neg(p \vee q) \vdash \neg p} \quad \neg\text{i}}{\neg(p \vee q) \vdash \neg p \wedge \neg q} \quad \wedge\text{i}}{\frac{\frac{\frac{\text{Ax}}{\neg(p \vee q), q \vdash q} \quad \text{vi,d}}{\neg(p \vee q), q \vdash p \vee q} \quad \text{Ax}}{\neg(p \vee q), q \vdash \perp} \quad \neg\text{i}}{\neg(p \vee q) \vdash \neg q} \quad \neg\text{i}}{\neg(p \vee q) \vdash \neg p \wedge \neg q} \quad \wedge\text{i}} \quad \neg\text{e}.$$

2.

$$\frac{\frac{\text{Ax}}{\neg p \vee \neg q, p \vee q \vdash p \vee q} \quad \frac{\frac{\frac{\text{Ax}}{\neg p \wedge \neg q, p \vee q, p \vdash p} \quad \frac{\frac{\text{Ax}}{\neg p \wedge \neg q, p \vee q, p \vdash \neg p} \quad \wedge\text{e,d}}{\neg p \wedge \neg q, p \vee q, p \vdash \perp} \quad \neg\text{e}}{\neg p \wedge \neg q, p \vee q, p \vdash \perp} \quad \neg\text{i}}{\neg p \wedge \neg q, p \vee q, q \vdash \perp} \quad \vee\text{e}}{\neg p \wedge \neg q, p \vee q, q \vdash \perp} \quad \vee\text{e}} \quad \neg\text{e}.$$

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4 Distributivités entre \wedge et \vee

1.

$$\frac{\frac{\frac{\text{Ax}}{p \wedge (q \vee r) \vdash p \wedge (q \vee r)} \quad \wedge\text{e,d}}{p \wedge (q \vee r) \vdash q \vee r} \quad \frac{\frac{\frac{\frac{\text{Ax}}{q, p \wedge (q \vee r) \vdash p \wedge (q \vee r)} \quad \wedge\text{e,g}}{p, p \wedge (q \vee r) \vdash p} \quad \frac{\frac{\text{Ax}}{p, q \wedge (q \vee r) \vdash q} \quad \wedge\text{i}}{q, p \wedge (q \vee r) \vdash p \vee q} \quad \text{vi,g}}{q, p \wedge (q \vee r) \vdash (q \vee r) \wedge (p \wedge r)} \quad \text{vi,g}}{p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)} \quad \vee\text{e}}{\frac{\frac{\frac{\frac{\text{Ax}}{r, p \wedge (q \vee r) \vdash p \wedge (q \vee r)} \quad \wedge\text{e,g}}{r, p \wedge (q \vee r) \vdash p} \quad \frac{\frac{\text{Ax}}{r, q \wedge (q \vee r) \vdash r} \quad \wedge\text{i}}{r, p \wedge (q \vee r) \vdash p \wedge r} \quad \text{vi,d}}{r, p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)} \quad \vee\text{e}}{p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)} \quad \vee\text{e}} \quad \wedge\text{i}.$$

2. On nomme $\varphi = (p \wedge q) \vee (p \wedge r)$.

$$\frac{\frac{\frac{\varphi \vdash (p \wedge q) \vee (p \wedge r)}{\varphi \vdash p} \text{Ax}}{\varphi \vdash p} \text{Ax} \quad \frac{\frac{\frac{\varphi, p \wedge q \vdash p \wedge q}{\varphi, p \wedge q \vdash p} \text{Ax}}{\varphi, p \wedge q \vdash p} \wedge \text{e,g} \quad \frac{\frac{\frac{\varphi, p \wedge r \vdash p \wedge r}{\varphi, p \wedge r \vdash p} \text{Ax}}{\varphi, p \wedge r \vdash p} \wedge \text{e,g}}{\varphi, p \wedge r \vdash p} \vee \text{e}}{\varphi \vdash p} \vee \text{e}}{\frac{\varphi \vdash p}{(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)} \wedge \text{i}} \quad \frac{\frac{\frac{\frac{\varphi, p \wedge q \vdash p \wedge q}{\varphi, p \wedge q \vdash q} \text{Ax}}{\varphi, p \wedge q \vdash q} \wedge \text{e,d} \quad \frac{\frac{\frac{\varphi, p \wedge r \vdash p \wedge r}{\varphi, p \wedge r \vdash r} \text{Ax}}{\varphi, p \wedge r \vdash r} \wedge \text{e,d}}{\varphi, p \wedge r \vdash r} \wedge \text{e,d}}{\varphi, p \wedge q \vdash q \vee r} \vee \text{i,g} \quad \frac{\frac{\frac{\varphi, p \wedge r \vdash p \wedge r}{\varphi, p \wedge r \vdash q \vee r} \text{Ax}}{\varphi, p \wedge r \vdash q \vee r} \wedge \text{e,d}}{\varphi, p \wedge r \vdash q \vee r} \vee \text{e}}{\varphi \vdash q \vee r} \vee \text{e}}{\frac{\varphi \vdash p}{\varphi \vdash q \vee r} \wedge \text{i}} \quad \frac{\varphi \vdash p \quad \varphi \vdash q \vee r}{(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)} \wedge \text{i}$$

5 Implications

1.

$$\frac{\frac{\frac{\frac{}{p, q \vdash q} \text{Ax}}{q \vdash p \rightarrow q} \rightarrow \text{i}}{q \vdash p \rightarrow q} \rightarrow \text{i}}{q \vdash p \rightarrow q} \rightarrow \text{i}}$$

2.

$$\frac{\frac{\frac{\frac{\frac{}{p, p \wedge q \vdash p \wedge q} \text{Ax}}{p, p \wedge q \vdash q} \wedge \text{e,d}}{p, p \wedge q \vdash q} \rightarrow \text{i}}{p \wedge q \vdash p \rightarrow q} \rightarrow \text{i}}{p \wedge q \vdash p \rightarrow q} \rightarrow \text{i}}$$

3.

$$\frac{\frac{\frac{\frac{\frac{}{p, p \rightarrow q \vdash p} \text{Ax}}{p, p \rightarrow q \vdash p} \text{Ax}}{p, p \rightarrow q \vdash p} \text{Ax}}{p, p \rightarrow q \vdash p} \text{Ax} \quad \frac{\frac{\frac{\frac{}{p, p \rightarrow q \vdash p \rightarrow q} \text{Ax}}{p, p \rightarrow q \vdash p \rightarrow q} \text{Ax}}{p, p \rightarrow q \vdash p \rightarrow q} \text{Ax}}{p, p \rightarrow q \vdash p \rightarrow q} \text{Ax}}{\frac{p, p \rightarrow q \vdash p \rightarrow q}{p, p \rightarrow q \vdash p} \wedge \text{i}} \rightarrow \text{e}}{\frac{p, p \rightarrow q \vdash p \rightarrow q}{p, p \rightarrow q \vdash p} \wedge \text{i}} \rightarrow \text{e}}{\frac{p, p \rightarrow q \vdash p \rightarrow q}{p \rightarrow q \vdash p \rightarrow (p \wedge q)} \rightarrow \text{i}}$$

4.

$$\frac{\frac{\frac{\frac{\frac{}{\neg q, p \rightarrow q, p \vdash p \rightarrow q} \text{Ax}}{\neg q, p \rightarrow q, p \vdash p \rightarrow q} \text{Ax}}{\neg q, p \rightarrow q, p \vdash p \rightarrow q} \text{Ax}}{\neg q, p \rightarrow q, p \vdash p \rightarrow q} \text{Ax} \quad \frac{\frac{\frac{\frac{}{\neg q, p \rightarrow q, p \vdash p} \text{Ax}}{\neg q, p \rightarrow q, p \vdash p} \text{Ax}}{\neg q, p \rightarrow q, p \vdash p} \text{Ax}}{\neg q, p \rightarrow q, p \vdash p} \text{Ax}}{\frac{\neg q, p \rightarrow q, p \vdash p \rightarrow q}{\neg q, p \rightarrow q, p \vdash p} \rightarrow \text{e}} \rightarrow \text{e}}{\frac{\neg q, p \rightarrow q, p \vdash p \rightarrow q}{\neg q, p \rightarrow q, p \vdash p} \rightarrow \text{e}} \rightarrow \text{e}}{\frac{\neg q, p \rightarrow q, p \vdash p \rightarrow q}{\neg q, p \rightarrow q, p \vdash \perp} \rightarrow \text{i}} \rightarrow \text{i}}{\frac{\neg q, p \rightarrow q, p \vdash \perp}{\neg q, p \rightarrow q \vdash \neg p} \rightarrow \text{i}} \rightarrow \text{i}}{\frac{\neg q, p \rightarrow q \vdash \neg p}{p \rightarrow q \vdash \neg q \rightarrow \neg p} \rightarrow \text{i}}$$

5.

$$\frac{\frac{\frac{\frac{\frac{}{p \wedge q, p \rightarrow r \vdash p \wedge q} \text{Ax}}{p \wedge q, p \rightarrow r \vdash p \wedge q} \text{Ax}}{p \wedge q, p \rightarrow r \vdash p \wedge q} \text{Ax}}{p \wedge q, p \rightarrow r \vdash p \wedge q} \text{Ax}}{\frac{p \wedge q, p \rightarrow r \vdash p \wedge q}{p \wedge q, p \rightarrow r \vdash p} \wedge \text{e,g}} \rightarrow \text{e}}{\frac{p \wedge q, p \rightarrow r \vdash p \wedge q}{p \wedge q, p \rightarrow r \vdash p \rightarrow r} \rightarrow \text{e}} \rightarrow \text{e}}{\frac{p \wedge q, p \rightarrow r \vdash p \wedge q}{p \rightarrow r \vdash (p \wedge q) \rightarrow r} \rightarrow \text{i}}$$

6.

$$\frac{\frac{\frac{\frac{\frac{}{p, q \vdash p} \text{Ax}}{p \vdash q \rightarrow p} \rightarrow \text{i}}{p \vdash q \rightarrow p} \rightarrow \text{i}}{\vdash p \rightarrow (q \rightarrow p)} \rightarrow \text{i}}{\vdash p \rightarrow (q \rightarrow p)} \rightarrow \text{i}}$$

7.

$$\frac{\frac{\frac{\frac{\frac{}{p \rightarrow q, p \vdash p \rightarrow q} \text{Ax}}{p \rightarrow q, p \vdash p \rightarrow q} \text{Ax}}{p \rightarrow q, p \vdash p \rightarrow q} \text{Ax}}{p \rightarrow q, p \vdash p \rightarrow q} \text{Ax}}{\frac{p \rightarrow q, p \vdash p \rightarrow q}{p \rightarrow q, p \vdash p} \rightarrow \text{e}} \rightarrow \text{e}}{\frac{p \rightarrow q, p \vdash p \rightarrow q}{p \rightarrow q, p \vdash p} \rightarrow \text{e}} \rightarrow \text{e}}{\frac{p \rightarrow q, p \vdash p \rightarrow q}{p \vdash (p \rightarrow q) \rightarrow q} \rightarrow \text{i}}$$

6 Implications (partie 1 : simplifications)

1.

$$\frac{\frac{\frac{}{p, p \rightarrow \neg p \vdash p} \text{Ax}}{p, p \rightarrow \neg p \vdash p} \text{Ax} \quad \frac{\frac{\frac{}{p, p \rightarrow \neg p \vdash p \rightarrow \neg p} \text{Ax}}{p, p \rightarrow \neg p \vdash p \rightarrow \neg p} \text{Ax}}{p, p \rightarrow \neg p \vdash \neg p} \rightarrow e}{p, p \rightarrow \neg p \vdash \neg p} \neg e}{\frac{p, p \rightarrow \neg p \vdash \perp}{p \rightarrow \neg p \vdash \neg p} \neg i} \neg i$$

2.

$$\frac{\frac{\frac{}{p, p \rightarrow q, \neg q \vdash p \rightarrow q} \text{Ax}}{p, p \rightarrow q, \neg q \vdash p \rightarrow q} \text{Ax} \quad \frac{\frac{\frac{}{p, p \rightarrow q, \neg q \vdash p} \text{Ax}}{p, p \rightarrow q, \neg q \vdash p} \text{Ax}}{p, p \rightarrow q, \neg q \vdash \neg q} \rightarrow e}{p, p \rightarrow q, \neg q \vdash \neg q} \neg e}{\frac{p, p \rightarrow q, \neg q \vdash \perp}{p \rightarrow q, \neg q \vdash \neg p} \neg i} \neg i$$

3.

$$\frac{\frac{\frac{}{p \rightarrow q, p \vee q} \text{Ax}}{p \rightarrow q, p \vee q} \text{Ax} \quad \frac{\frac{\frac{}{p \rightarrow q, p \vee q, p \vdash p \rightarrow q} \text{Ax}}{p \rightarrow q, p \vee q, p \vdash p \rightarrow q} \text{Ax}}{p \rightarrow q, p \vee q, p \vdash q} \rightarrow e}{p \rightarrow q, p \vee q, p \vdash q} \rightarrow e}{\frac{p \rightarrow q, p \vee q, q \vdash q} \vee e} \vee e$$

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4.

$$\frac{\frac{\frac{}{p, p \rightarrow q, p \rightarrow \neg q \vdash p \rightarrow q} \text{Ax}}{p, p \rightarrow q, p \rightarrow \neg q \vdash p \rightarrow q} \text{Ax} \quad \frac{\frac{\frac{}{p, p \rightarrow q, p \rightarrow \neg q \vdash p} \text{Ax}}{p, p \rightarrow q, p \rightarrow \neg q \vdash p} \text{Ax}}{p, p \rightarrow q, p \rightarrow \neg q \vdash q} \rightarrow e}{p, p \rightarrow q, p \rightarrow \neg q \vdash q} \rightarrow e}{\frac{p, p \rightarrow q, p \rightarrow \neg q \vdash \perp}{p \rightarrow q, p \rightarrow \neg q \vdash \neg p} \neg i} \neg i$$

5. On pose $\Gamma = p, p \rightarrow (q \vee r), \neg q, \neg r, q \vee r$.

$$\frac{\frac{\frac{}{p, p \rightarrow (q \vee r), \neg q, \neg r \vdash p \rightarrow (q \vee r)} \text{Ax}}{p, p \rightarrow (q \vee r), \neg q, \neg r \vdash p \rightarrow (q \vee r)} \text{Ax} \quad \frac{\frac{\frac{}{p, p \rightarrow (q \vee r), \neg q, \neg r \vdash p} \text{Ax}}{p, p \rightarrow (q \vee r), \neg q, \neg r \vdash p} \text{Ax}}{p, p \rightarrow (q \vee r), \neg q, \neg r \vdash q \vee r} \rightarrow e}{p, p \rightarrow (q \vee r), \neg q, \neg r \vdash q \vee r} \rightarrow e}{\frac{p, p \rightarrow (q \vee r), \neg q, \neg r \vdash \perp}{p \rightarrow (q \vee r), \neg q, \neg r \vdash \neg p} \neg i} \neg i$$

6. On pose $\Gamma = p \rightarrow (q \rightarrow r), p, \neg r, q$.

$$\frac{\frac{\frac{\overline{\Gamma \vdash p \rightarrow (q \rightarrow r)} \text{ Ax}}{\Gamma \vdash q \rightarrow r}}{\Gamma \vdash r} \rightarrow e \quad \frac{\overline{\Gamma \vdash p} \text{ Ax}}{\Gamma \vdash q} \rightarrow e}{\Gamma \vdash \perp} \rightarrow e \quad \frac{\overline{\Gamma \vdash \neg r} \text{ Ax}}{\Gamma \vdash \perp} \neg e}{\frac{\Gamma \vdash \perp}{p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q} \rightarrow i} \neg i$$

7. On pose $\Gamma = p \rightarrow (q \rightarrow r), p, \neg r, q$.

$$\frac{\frac{\frac{\overline{\Gamma \vdash p \rightarrow (q \rightarrow r)} \text{ Ax}}{\Gamma \vdash q \rightarrow r}}{\Gamma \vdash r} \rightarrow e \quad \frac{\overline{\Gamma \vdash p} \text{ Ax}}{\Gamma \vdash q} \rightarrow e}{\Gamma \vdash \perp} \rightarrow e \quad \frac{\overline{\Gamma \vdash \neg r} \text{ Ax}}{\Gamma \vdash \perp} \neg e}{\frac{\Gamma \vdash \perp}{p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q} \rightarrow i} \rightarrow i$$

8. On pose $\Gamma = q, p \rightarrow (q \rightarrow r), q \rightarrow p$.

$$\frac{\frac{\frac{\overline{\Gamma \vdash p \rightarrow (q \rightarrow r)} \text{ Ax}}{\Gamma \vdash q \rightarrow r} \quad \frac{\overline{\Gamma \vdash q \rightarrow p} \text{ Ax}}{\Gamma \vdash p} \rightarrow e \quad \frac{\overline{\Gamma \vdash q} \text{ Ax}}{\Gamma \vdash q} \rightarrow e}{\Gamma \vdash r} \rightarrow e}{\Gamma \vdash r} \rightarrow e \quad \frac{\Gamma \vdash r}{p \rightarrow (q \rightarrow r), q \rightarrow p, \vdash q \rightarrow r} \rightarrow i}{\frac{\Gamma \vdash r}{p \rightarrow (q \rightarrow r), q \rightarrow p, \vdash q \rightarrow r} \rightarrow i} \rightarrow i$$

9.

$$\frac{\frac{\frac{\overline{p \rightarrow (p \rightarrow q), p \vdash p \rightarrow (p \rightarrow q)} \text{ Ax}}{p \rightarrow (p \rightarrow q), p \vdash p \rightarrow q} \rightarrow e \quad \frac{\overline{p \rightarrow (p \rightarrow q), p \vdash p} \text{ Ax}}{p \rightarrow (p \rightarrow q), p \vdash p} \rightarrow e}{p \rightarrow (p \rightarrow q), p \vdash p} \rightarrow e \quad \frac{\overline{p \rightarrow (p \rightarrow q), p \vdash p} \text{ Ax}}{p \rightarrow (p \rightarrow q), p \vdash p} \rightarrow e}{\frac{p \rightarrow (p \rightarrow q), p \vdash p}{p \rightarrow (p \rightarrow q), p \vdash q} \rightarrow e} \rightarrow e$$

7 Implications (partie 2 : transformations)

1. *c.f.* Exercice 5, question 4.

2.

$$\frac{\frac{\frac{}{p \rightarrow q, p \wedge r \vdash p \rightarrow q} \text{Ax}}{p \rightarrow q, p \wedge r \vdash q} \text{Ax} \quad \frac{\frac{\frac{}{p \rightarrow q, p \wedge r \vdash p \wedge r} \text{Ax}}{p \rightarrow q, p \wedge r \vdash p} \wedge e, g}{p \rightarrow q, p \wedge r \vdash r} \wedge i}{p \rightarrow q, p \wedge r \vdash q \wedge r} \wedge i}{p \rightarrow q \vdash (p \wedge q) \rightarrow (q \wedge r)} \rightarrow i$$

3.

$$\frac{\frac{\frac{}{(p \wedge r) \rightarrow (q \wedge r), r, p \vdash (p \wedge r) \rightarrow (q \wedge r)} \text{Ax}}{(p \wedge r) \rightarrow (q \wedge r), r, p \vdash p \wedge r} \text{Ax}}{(p \wedge r) \rightarrow (q \wedge r), r, p \vdash q \wedge r} \wedge i}{(p \wedge r) \rightarrow (q \wedge r), r, p \vdash q \wedge r} \wedge i}{(p \wedge r) \rightarrow (q \wedge r), r, p \vdash q} \vee e, g}{(p \wedge r) \rightarrow (q \wedge r), r \vdash p \wedge q} \rightarrow i$$

4.

$$\frac{\frac{\frac{\frac{}{p \rightarrow q, p \vee r, p \vdash p \rightarrow q} \text{Ax}}{p \rightarrow q, p \vee r, p \vdash q} \text{Ax}}{p \rightarrow q, p \vee r, p \vdash q \vee r} \vee i, g}{p \rightarrow q, p \vee r \vdash q \vee r} \rightarrow i}{p \rightarrow q \vdash (p \vee r) \rightarrow (q \vee r)} \rightarrow i$$

5. On pose $\Gamma = p, q, (p \wedge q) \rightarrow r, \neg r$.

$$\frac{\frac{\frac{\frac{}{\Gamma \vdash p} \text{Ax} \quad \frac{}{\Gamma \vdash q} \text{Ax}}{\Gamma \vdash p \wedge q} \wedge i}{\Gamma \vdash r} \rightarrow e}{\Gamma \vdash \perp} \text{Ax}}{\neg r, p, (p \wedge q) \rightarrow r \vdash \neg q} \neg i}{(p \wedge q) \rightarrow r, \neg r, \vdash p \rightarrow \neg q} \rightarrow i$$

6. On pose $\Gamma =$