— INTERNSHIP REPORT —

CLASSIFYING COVERING SPACES IN HOMOTOPY TYPE THEORY [a]

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Contents

1 Introduction.

Algebraic topology is the study of topological spaces through algebra. We may compare two spaces by studying *invariants*. A simple invariant is the number of connected components in the space.

Classifying spaces "*up to equality*" is very restrictive. When we consider the usual example of the coffee mug that can be deformed into a torus, we want to consider that the two spaces are identical. This tells us that we should consider spaces "*up to continuous deformation*," we will give a more precise definition in section 2.

The goal of this internship was to formally prove (*i.e.* in a proof assistant) a classical result in algebraic topology: there is a one to one correspondance between covering spaces and subgroups of the fundamental group. All of the definitions will be give in section 2. However, the main idea is: by studying subgroups, we get geometric insights.

To prove this result, we use *Homotopy Type Theory* (*HoTT*). This framework links *geometry* and *logic* in one coherent paradigm based on *Type Theory* (used in most proof assistants). It allows us to very simply reason about many of the key concepts of algebraic topology. Instead of thinking of types as logical propositions (as it usual in Type Theory), we think of them as topological spaces. We will discuss more about HoTT in section 3.

Agda is the proof assistant used during this internship. When using proof assistants like Rocq or Lean, we expect to use tactics in order to prove a result (so, a type, as it is usual in Type Theory). These tactics generate a term of the corresponding type. However, Agda doesn't go in this direction. Instead, we write terms directly with a Haskell-like syntax. The *Cubical* version of Agda allows us to use *Cubical Homotopy Type Theory*, a variant of the one presented in the *HoTT Book* [Uni13] (the main reference for Homotopy Type Theory). A more complete presentation of Agda will be given in section 4.

2 Some elements of Algebraic Topology.

In this section, we will give an introduction to algebraic topology. The main source for this section is [Hat02, Chapters 1 & 2].

In this section, we will write $\mathbf{I}$ for $[0, 1]$–the unit interval. Let X be a topological space, that is, some space with a notion of *openness* and thus *continuity*.

2.1 Paths and Loops.

Let us start with a definition of paths and loops.

Definition 1.
- ▹ A ***path*** from x to y (in the topological space X) is a continuous map $p : \mathbf{I} \to X$ such that $p(0) = x$ and $p(1) = y$.
- ▹ A ***loop*** around x is a path from x to x.

We will write $p : x \rightsquigarrow y$ when p is a path from x to y (in some implied space X). We expect to be able to *concatenate* paths, and also to *reverse* them.

Definition 2. Let $p : x \rightsquigarrow y$ and $q : y \rightsquigarrow z$ be paths.

- ▹ The ***reversed*** or ***inverse*** of path p is the path p^{-1} defined by:

$$p^{-1}(t) := p(1 - t).$$

- ▹ The ***concatenation*** of paths p and q is the path $p \cdot q$ defined by :

$$p \cdot q(t) := \begin{cases} p(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ q(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

- ▹ The ***constant loop*** at point x is the loop refl_x de-

fined by:
$$\mathsf{refl}_x(t) := x.$$

We can take the reverse of any paths but we need that the common endpoint matches when we want to concatenate paths. The notation refl_x may seem unusual when talking about paths, but it'll make more sense after section .

We'd love to have some nice properties with the concatenation and inverse: for example, associativity of concatenation, or that refl_x behaves like a neutral element for concatenation. However, *in general*, we do not have
$$(p \cdot q) \cdot r \neq p \cdot (q \cdot r).$$

This is a *timing issue*. For the first path, we spend the first quarter travelling along p, then the next quarter along q, and finally the rest along r. For the second path, we spend the first half travelling along p and then a quarter along q and finally the rest along r.

Another example is $p \cdot p^{-1}$: this path "behaves" like the loop refl_x, but it's not equal to refl_x. And for this example, it's more than a timing issue, we need to *smoothly deform* the path p, bringing closer the endpoint y to x until we get refl_x.

Strict equality is not the right way to compare paths. The right way is *equality up to smooth deformation*, also known as a *homotopy*.

Definition 3. A ***homotopy*** from a path $p : x \rightsquigarrow y$ to a path $q : x \rightsquigarrow y$ is a continuous map
$$h : \mathbf{I} \times \mathbf{I} \to X,$$
such that: for all $t \in \mathbf{I}$,

1. $h(0,t) = p(t)$,
2. $h(1,t) = q(t)$,
3. $h(t,0) = x$,
4. $h(t,1) = y$.

We say that p and q are ***homotopic***, written $p \approx_{\mathrm{p}} q$ when there exists a homotopy from p to q.

In the definition above, the conditions are *boundary conditions*: that is, the initial slice (when the first parameter is 0) is p, the final slice is q and all the slices are paths from x to y. The relation "homotopic to" is an equivalence relation.

Uncurrying the definition of a homotopy, we have, in some sense, a path $h : p \rightsquigarrow q$ in the space of paths between x and y. To be very formal, we'd need to define a topology on the set of paths between x and y, which we don't want to do. However, it is an important idea to keep in mind.

Now that we have defined a new way of comparing paths, we need to make sure that

1. the notions of concatenation and inverse are defined up to homotopy,
2. and we get the nice properties we expect from concatenation and inverses.

Luckily, we do.

Lemma 1. Let $p, q : u \rightsquigarrow v$ and $r, s : v \rightsquigarrow v$ and $t : w \rightsquigarrow x$. We have:

- if $p \approx_{\mathrm{p}} q$ then $p^{-1} \approx_{\mathrm{p}} q^{-1}$;
- if $p \approx_{\mathrm{p}} q$ and $r \approx_{\mathrm{p}} s$ then $p \cdot r \approx_{\mathrm{p}} q \cdot s$;
- $(p \cdot r) \cdot t \approx_{\mathrm{p}} p \cdot (r \cdot t)$;
- $p \cdot \mathsf{refl}_y \approx_{\mathrm{p}} p \approx_{\mathrm{p}} \mathsf{refl}_x \cdot p$;
- $p \cdot p^{-1} \approx_{\mathrm{p}} \mathsf{refl}_x$;
- $p^{-1} \cdot p \approx_{\mathrm{p}} \mathsf{refl}_y$. □

With these property in mind, we can now introduce the fundamental group.

2.2 Fundamental Group.

Definition 4. Given a topological space X with $x \in X$ a point (we say that (X, x) is a ***pointed topological space***), its ***fundamental group*** $\pi_1(X, x)$ is the set of homotopy classes of loops at x in space X.

The fundamental group really has the structure of a group, thanks to lemma 1.

The fundamental group is one of those *invariants* discussed in the introduction. Let us give a simple example of how the fundamental group can be used to compare topological spaces.

Example 1. Consider the space $\mathbb{R}^2$ with the usual topology. The group $\pi_1(\mathbb{R}^2, (1,0))$ is trivial: any loop is homotopic to $\mathsf{refl}_{(1,0)}$. You can think of scaling any loop at $(1,0)$ until it vanishes into a constant loop. As we didn't tear the loop, our deformation was smooth, *i.e.*, a homotopy.

Example 2. Consider the subspace

$$V := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1\}$$

of $\mathbb{R}^2$. It corresponds to $\mathbb{R}^2$ without a unit open disk. We have that $\pi_1(V, (1, 0))$ is isomorphic to $\mathbb{Z}$. Let us see why in 3 steps.

1. There is a loop at point $(1, 0)$ that goes once "around the hole in V": it is the loop p defined by $p(t) := (\cos 2\pi t, \sin 2\pi t)$.
2. Any two different powers[b] of loop p are non homotopic. For example, p is not homotopic to the constant $\mathsf{refl}_{(1,0)}$ as it'd require cutting p to "go through" the hole in V.
3. Given any loop at $(1, 0)$, we can project it into a loop such that every point is at distance 1 from the origin. Then, we can smoothly deform the loop into p^n (where n is the *signed* number of times you go around the hole).

This tells us that $\pi_1(V, (1, 0))$ is freely generated by the homotopy class of p and thus is isomorphic to $\mathbb{Z}$.

The two spaces $\mathbb{R}^2$ and V are topologically different: they have a different fundamental group. It makes sense because $\mathbb{R}^2$ is topologically equivalent to a single point, and V is topologically equivalent to the unit circle $\mathbb{S}^1$. Only studying the number of connected components wouldn't have given us this geometric insight.

Change of base point. The ***base point*** of a pointed space (X, x) is x. What happens to the fundamental group when we change the base point? The answer is: not a lot, as long as the two base points have a path between them.

Proposition 1. Let $p : x \rightsquigarrow y$ be a path. Then,

$$\begin{aligned} \pi_1(X, x) &\longrightarrow \pi_1(X, y) \\ [\, q \,] &\longmapsto [\, p^{-1} \cdot q \cdot p \,] \end{aligned}$$

is a group isomorphism, where $[\, q \,]$ is the homotopy class of a path q. □

So, when X is non-empty and ***path-connected*** (that is, for any two pairs of points, there is a path between them), there is no need to specify a base point.

When there is no path between x and y, the fundamental groups could be *very different*. For example, if we consider embedding $\mathbb{R}^2$ and V on two parallel planes in $\mathbb{R}^3$ and considering the union of those two planes, then the fundamental group in the $\mathbb{R}^2$-plane is still trivial and other one is still isomorphic to $\mathbb{Z}$.

Functoriality. The π_1 "operation" acts on pointed spaces, but also acts on *pointed continuous maps*: that is, a base point-preserving continuous map between pointed spaces.[c] We say that π_1 is ***functorial***.

Proposition 2. Let (X, x) and (Y, y) be pointed spaces. A pointed continuous map

$$f : (X, x) \to (Y, y)$$

induces a group homomorphism

$$\begin{aligned} \pi_1(f) : \pi_1(X, x) &\longrightarrow \pi_1(Y, y) \\ [\, p \,] &\longmapsto [\, f \circ p \,]. \end{aligned}$$

Thus, a pointed ***homeomorphism*** (a bijective continuous map whose inverse is also continuous) induces a group isomorphism.

Homeomorphic spaces have the same fundamental group. It would seem reasonable to compare topological spaces *up to homeomorphism* but, as with path equality, it is too restrictive. For example, $\mathbb{R}^2$ and $\{0\}$ are topologically equivalent, but aren't homeomorphic (simply for cardinality reasons). We need to use *homotopy equivalences*.

2.3 Homotopy equivalence.

Let us first define a *homotopy of maps*.

Definition 5. A ***homotopy (of maps)*** from a continuous map $f : X \to Y$ to a continuous map $g : X \to Y$ is a continuous map

$$h : X \times \mathbf{I} \to Y,$$

such that: for all $x \in X$,

[b] That is, repeated concatenation of p, or p^{-1} if the power is negative, as usual when manipulating groups.

[c] More explicitly, $f : (X, x) \to (Y, y)$ is a ***pointed continuous map*** if $f : X \to Y$ is continuous and if $f(x) = y$.

1. $h(x,0) = f(x)$,
2. $h(x,1) = g(x)$.

When f and g are ***map-homotopic*** (that is, there exists a homotopy of maps between them), we write $f \approx g$. It is an equivalence relation.

We already introduced some notation for homotopies of paths: $p \approx_{\mathrm{p}} q$. Is it the same as $p \approx q$? No, in path homotopy, we require that the endpoints of all slices are x and y. For example, any path p is map-homotopic to the constant loop refl_x.

Now that we have a new way of comparing functions, we need to make sure that composition and inverses (when they exist) are compatible with map-homotopy.

Lemma 2. Given $f, u : X \to Y$ and $g, v : Y \to Z$, such that $f \approx u$ and $g \approx v$, we have that:

- $f \circ g \approx u \circ v$;
- $f^{-1} \approx u^{-1}$ if f and u are homeomorphisms.

Then we can finally define the notion of *homotopy equivalences*.

Definition 6. A ***homotopy equivalence*** between X and Y is a two maps $f : X \to Y$ and $g : Y \to X$ such that:

$$f \circ g \approx \mathrm{id}_Y \qquad \text{and} \qquad g \circ f \approx \mathrm{id}_X.$$

We write $X \simeq Y$ when X and Y are ***homotopy equivalent***. It is an equivalence relation.

Example 3. The spaces $\mathbb{R}^2$ and $\{0\}$ are homotopy equivalent. We define:

- $f : \mathbb{R}^2 \to \{0\}, x \mapsto 0$;
- $g : \{0\} \to \mathbb{R}^2, 0 \mapsto (0,0)$.

On the one hand, we easily have that $f \circ g = \mathrm{id}_{\{0\}}$. On the other hand, we need to provide a homotopy between $x \in \mathbb{R}^2 \mapsto (0,0) \in \mathbb{R}^2$ and $\mathrm{id}_{\mathbb{R}^2}$, so a continuous map

$$h : \mathbb{R}^2 \times \mathbf{I} \to \mathbb{R}^2,$$

such that the initial slice is $x \mapsto (0,0)$ and the final one is $x \mapsto x$. We can define it, for example, with

$$h((x,y),t) := (tx, ty).$$

This can be generalized to $\mathbb{R}^n \simeq \{0\}$ for any $n \in \mathbb{N}$.

Example 4. Recalling the definition of V from example 2:

$$V := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1\}.$$

Then V is homotopy equivalent to $\mathbb{S}^1$. Here is a homotopy equivalence:

- the map $f : \mathbb{S}^1 \to V$ is the inclusion map;
- the map $g : V \to \mathbb{S}^1$ is $\vec{v} \mapsto \vec{v}/||\vec{v}||$;
- we have $g \circ f = \mathrm{id}_{\mathbb{S}^1}$;
- we provide $h : (\vec{v}, t) \mapsto (1-t)\vec{v} + t\vec{v}/||\vec{v}||$ a homotopy between $f \circ g : \vec{v} \to \vec{v}/||\vec{v}||$ and id_V.

To justify the use of homotopy equivalences, we should give some results about homotopy equivalences and the fundamental group.

Proposition 3. If $f : (X,x) \to (Y,y)$ is a pointed homotopy equivalence, then the induced group homomorphism $\pi_1(f) : \pi_1(X,x) \to \pi_1(Y,y)$ is an isomorphism. □

This concludes the subsection on homotopy equivalences.

2.4 Covering spaces.

We will start by defining what a *covering space* is and then give some examples.

Definition 7. Let X be a topological space. A ***covering space*** is a space $\tilde{X}$ with a map $p : \tilde{X} \to X$ with the following property: every point $\tilde{x} \in \tilde{X}$ has an open neighborhood U whose preimage $p^{-1}(U)$ is homeomorphic to a space of the form $U \times D$ where D is a discrete space.[d]

The copies of U in the preimage $p^{-1}(U)$ are called ***sheets*** of $\tilde{X}$ over U.

explain why we care about covering spaces.

Here are some examples of covering spaces.

Example 5. Given a topological space X and some discrete space D, the space $\tilde{X} := X \times D$ with the map $p : (x,d) \mapsto x$ is called a ***trivial covering space*** of X.

Example 6. For any $n \in \mathbb{N}$, we can construct a covering space of $\mathbb{S}^1$: we choose $\tilde{X}$ to be $\mathbb{S}^1$, which we

[d] This means a set with the discrete topology: for example sets of the form $[\![1,n]\!]$ or $\mathbb{Z}$ are discrete spaces.

think as the subset of $\mathbb{C}$ whose points have a magnitude of 1, and we choose the map

$$\begin{aligned} p_n : \mathbb{S}^1 &\longrightarrow \mathbb{S}^1 \\ z &\longmapsto z^n. \end{aligned}$$

This corresponds to a covering of $\mathbb{S}^1$ with n sheets. Figure 1 shows this covering space for $n = 4$ (left one).

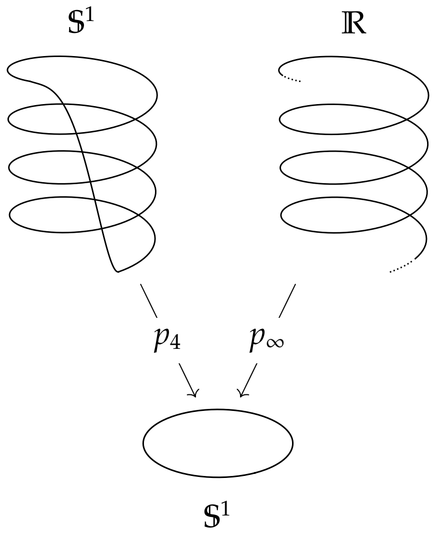

Figure 1 | *4-sheet cover and universal cover of* $\mathbb{S}^1$

Figure 1 shows a smooth deformation of $\tilde{X}$ such that applying p corresponds to "flattening" that space on the unit circle $\mathbb{S}^1$.

Example 7. Another covering space of $\mathbb{S}^1$ is $\mathbb{R}$ with the map $p_\infty : x \mapsto e^{2i\pi x}$. We call it the *universal cover* of $\mathbb{S}^1$ for reasons we will explain later. Figure 1 shows this covering space (right one).

what other example? $\mathbb{S}^1 \vee \mathbb{S}^1$ or $\mathbb{T}^2$?

We can define *morphisms between covering spaces*. Thus, for a given topological space X, the collection of covering spaces of X with morphisms between them forms a ***category*** (*i.e.* composition of morphisms of cover and identities behave as we expect[e]).

Definition 8. Given $(\tilde{X}, \tilde{p})$ and $(\bar{X}, \bar{p})$ two covering spaces of X, a ***morphism of covering spaces*** from $(\tilde{X}, \tilde{p})$ to $(\bar{X}, \bar{p})$ is a continuous map $f : \tilde{X} \to \bar{X}$ such that the following diagram commutes:

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "\tilde{p}"'] & & \bar{X} \arrow[dl, "\bar{p}"] \\ & X & \end{tikzcd}$$

i.e. such that $\bar{p} \circ f = \tilde{p}$.

We can now define what *universal covering* means (and fully understand example 7).

Definition 9. A covering $(\tilde{X}, \tilde{p})$ of X is said to be ***universal*** if for any covering $(\bar{X}, \bar{p})$ of X there is a morphism from $(\tilde{X}, \tilde{p})$ to $(\bar{X}, \bar{p})$.

We can safely say "*the*" universal cover as it is unique up to isomorphism.

2.5 The Galois Correspondance.

3 Homotopy Type Theory.

4 Cubical Agda.

5 Conclusion.

References

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[e] That is: associativity of composition and identity as neutral elements of composition.

Todo list