TD n°1 Modelization & linear Programming

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Exercise 1. Lumberjack.

Q1.
$$\begin{cases} w + p \leq 100 \\ 10w + 50p \leq 4000 \\ w, p \geq 0 \end{cases}$$

maximize $50w + 120p =: y$

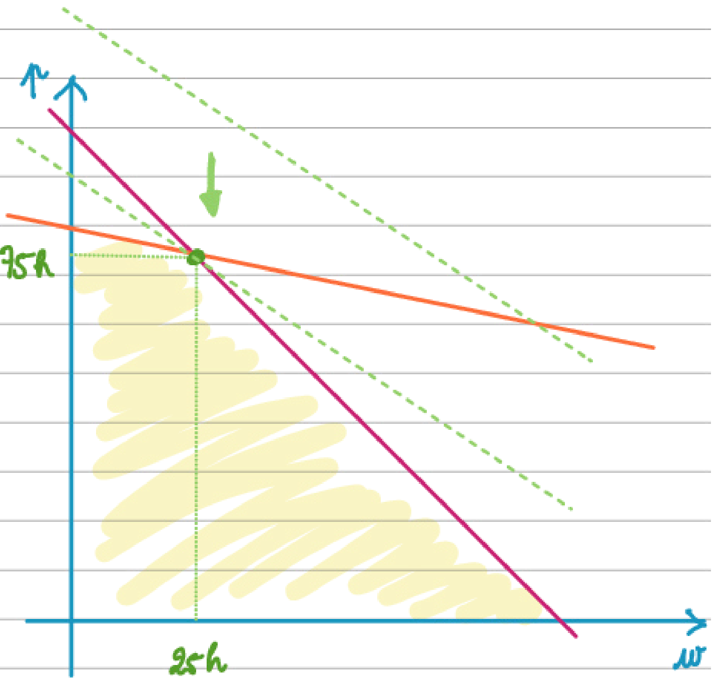
$\Updownarrow$

find a line

$$p = \frac{y}{120} - \frac{50}{120} a$$

with the highest y-intercept.

Q2.



Q3. The best strategy is 25h with the wood cut and 75h re-seeded, for a profit of 10 250 k\$.

Exercise 2. Student diet problem

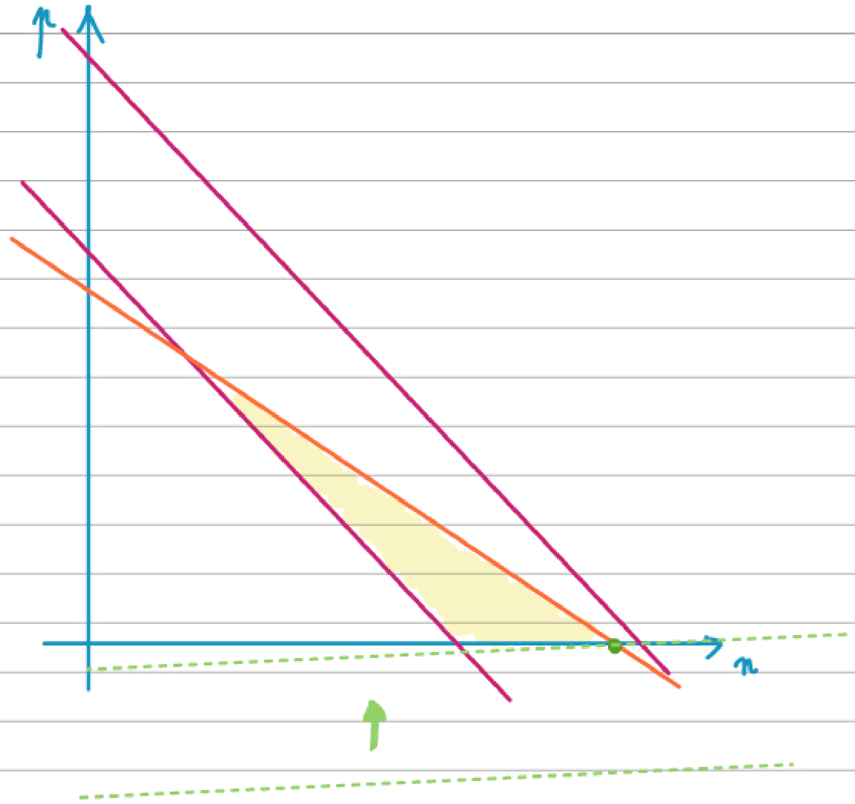
Q1.
$$\begin{cases} 460n + 700p + 400r \leq 3500 \\ 460n + 700p + 400r \geq 2500 \quad (1) \\ 21n + 20p \leq 120 \\ 21n + 20p \geq 80 \quad (2) \\ n, p, r \geq 0 \end{cases}$$

minimize $0{,}5n + 3{,}5p + 0{,}5r$

Q2. Instead of drawing in 3D, as we are minimizing a strictly increasing function, we consider two cases: if (1) is an equality or if (2) is an equality.

Case 1: $460n + 700p + 400r = 2500$

$\iff r = 6{,}25 - 1{,}75p - 1{,}15n$



Dans ce cas, l'optimal est $n = 125/23$, $p = 0$ et $r = 0$ (*)

avec un prix de 2,92 € environ.

Dans l'autre cas, on trouve $n = \frac{80}{21}$, $p = 0$ et $r = 25/4$, avec un prix de 5,03€ environ. (**).

On conclut que (*) est l'optimal.

Exercice 3. Bank allocation.

c : crédit client, v : crédit voiture p : prêt.

$$\begin{aligned} v + p &\geq 60\% \times (v+p+c) \Rightarrow \\ p &\geq 40\% \ (v+p+c) \Rightarrow \\ 6\% c + 4\% v + 2\% p &\leq (v+p+c)\, 3{,}2\% \Rightarrow \end{aligned}
\quad
\begin{cases} v + c + p \leq 10^6 \\ 0{,}3 v + 0{,}3 p - 0{,}6 c \geq 0 \\ 0{,}6 p - 0{,}4 v - 0{,}4 p \geq 0 \\ 0{,}028 c + 0{,}0008 v - 0{,}0012 p \leq 0 \\ v, p, c \leq 0 \end{cases}$$

Maximize $0{,}06 c + 0{,}04 v + 0{,}02 p$

We can compute the solution by case by case analysis.

Exercise 6. Independant Set Problem

Variables : x_v for every vertex $v \in V$

Constraints : for any edge $uv \in E$, $x_u + x_v \leq 1$

for any vertex $v \in V$, $1 \geq x_v \geq 0$

Maximize $\sum_{v \in V} x_v$ = size of the independant set

Exercise 7. Dominating Set Problem

Variables x_u $u \in V$

Constraints $\forall u \in V$, $0 \leq x_u \leq 1$

$\forall u \in V$, $\sum_{v \in N[u]} x_v \geq 1$

↳ closed neighborhood

Minimize $\sum_{u \in V} x_u$

Exercise 8. N-queens problem

Variables : $x_{i,j}$, $i,j \in [\![1, N]\!]$

Constraints : $\forall i$, $\sum x_{i,j} \leq 1$

$\sum x_{j,i} \leq 1$

$\sum x_{i,i-j} \leq 1$

$\sum x_{i,i+j} \leq 1$

$\forall i,j$ $0 \leq x_{i,j} \leq 1$

Maximize $\sum_{ij} x_{ij}$

TD n° 2 Linear Programming and the simplex method

Exercise 1. Computer production

Q1.

$$\begin{cases} n + d \leq 10000 \\ n + 2d \leq 15000 \\ 4n + 3d \leq 38000 \\ n, d \geq 0 \end{cases}$$

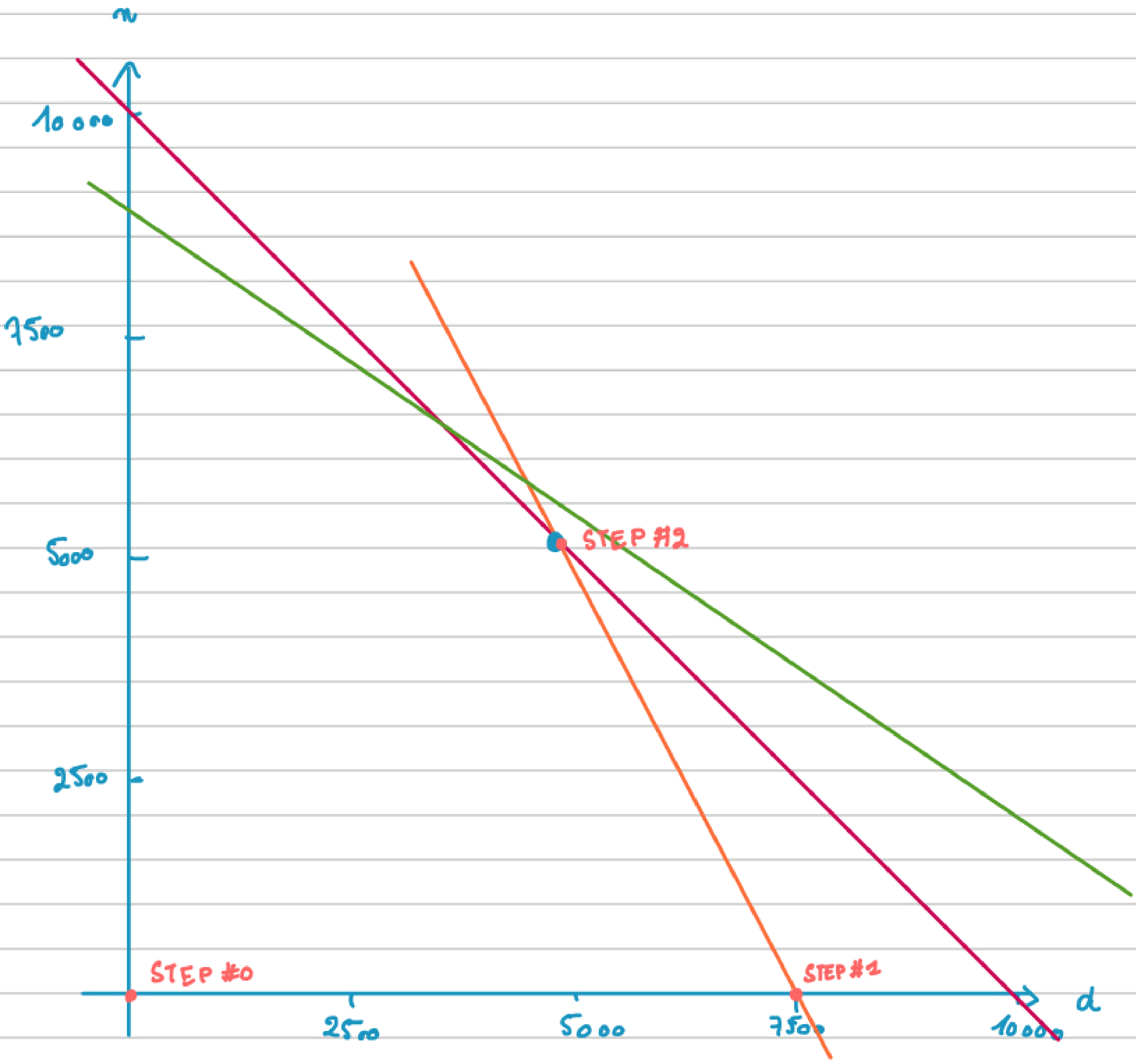
$$n = 9500 - \frac{3}{4} d$$

$$\frac{3}{4} d = 9500$$

max $750 n + 1000 d$

Q2.

L'optimal est $n = 5000$ et $d = 5000$.



$$\begin{cases} u = 10 - n - d \\ v = 15 - n \cdot 2d \\ w = 38 - 4n - 3d \\ z = 75n + 100d \end{cases}$$

$\boxed{\text{STEP \#1}}$ Choice for pivot : d enters and v leaves.

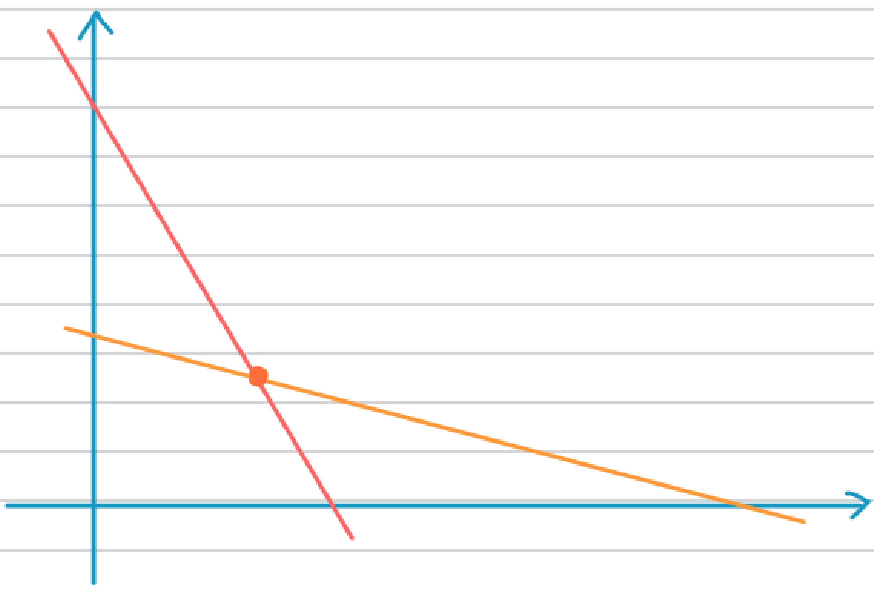
We have $d = 7.5 - \frac{n}{2} - \frac{v}{2}$ thus, we substitute

$$\begin{cases} u = 2,5 - \frac{n}{2} + \frac{v}{2} \\ d = 7,5 - \frac{n}{2} - \frac{v}{2} \\ w = 15,5 - 2,5n - 2,5v \\ z = 750 + 25n - 50v \end{cases}$$

$\boxed{\text{STEP \#2}}$ Choice for pivot : n enters and u leaves

Solution: $5000 = n = d$ for a reward of 87500

Exercise 2. Multiple optimal solutions



TD n° 4

Exercice 1. Le cas non faisable / non faisable.

Dual \ Primal	Empty domain	Unbounded	Optimal solution
Empty domain	Yes	Yes	No
Unbounded	Yes	No	No
Optimal solution	No	No	Yes

Considérons $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \leq \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\max x+y$

Son dual est $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \geq \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\min x+y$

Exercice 2.

Q1. $(P)\begin{cases} \max \sum c_{ij} x_{ij} \\ (a_i) : \sum_{j=1}^{4} x_{ij} \leq 1 \\ (t_j) : \sum_{i=1}^{4} x_{ij} \leq 1 \end{cases}$

Solution optimale : $x_{13} = x_{24} = x_{32} = x_{41} = 1$.

pour une affectation agent compétence de 20

Q2. We compute $4 \times (a_1) + 4 \times (a_2) + 5 \times (a_3) + 3 \times (a_4)$
$+ (t_1) + (t_2) + 2 \cdot (t_4)$
and we obtain that

$$\sum \begin{array}{c|c} \begin{array}{|c|c|c|c|} \hline 25 & 15 & 16 & 30 \\ \hline 20 & 25 & 16 & 36 \\ \hline 30 & 36 & 25 & 49 \\ \hline 16 & 16 & 9 & 25 \\ \hline \end{array} & \begin{array}{c} 4 \\ 4 \\ 5 \\ 3 \end{array} \\ \begin{array}{cccc} 1 & 1 & 0 & 2 \end{array} & \end{array} \leq 20$$

Exercise 3.

Q1. A vertex v is a point of P that ~~is~~ cannot be written as $\lambda v_1 + (1-\lambda) v_2 = v$ for some $\lambda \in [0,1]$, $x_1, x_2 \in P$. Such values (λ, x_1, x_2) provide an easy certificate.

Q2. Define $\varepsilon := \min\left(\{\min(x_i, \frac{1}{2} - x_i) \mid i \in I^-\} \cup \{\min(1-x_i, x_i - \frac{1}{2}) \mid i \in I^+\}\right)$

Then it is easy to see that $x + \varepsilon y$ is a point of P.

Q3. We have that $v^+ := x + \varepsilon y \in P$ and $v^- := x - \varepsilon y \in P$

but $\frac{1}{2} v^+ + \frac{1}{2} v^- = v$ thus it is only possible if ε is undefined.

Q4. We solve the LP problem :

$$\begin{array}{|l} \min \sum_{i \in V} c(i) x_i \\ \text{such that } \forall ij \in E, \ x_i + x_j \leq 1. \\ \qquad \forall i \in V, \ x_i \geq 0 \end{array}$$

Take the solution and put $\frac{1}{2} \to 1$.

Exercise 4.

Q1. (P)

$$\begin{array}{|l} \max 0 \\ \text{such that} \\ \forall v \in V, \ w_v \geq 0 \\ \forall v \in V, \ \sum_{uv \in E} w_u - \sum_{vu \in E} w_u \geq 0 \\ \sum_{v \in V} w_v = 1. \end{array} \quad \left. \begin{array}{l} Ax \leq 0 \\ x \geq 0 \\ \sum x = 1 \end{array} \right)$$

Q2. (D)

$$\begin{array}{|l} \min \ z \\ \text{such that} \\ \qquad y \geq 0 \\ \qquad -Ay + z\mathbb{1} \geq 0 \end{array}$$

Q3 0 is a solution of (D)

thus (P) is non-empty.

Exercise 5.

Q1. Consider the instance $T_1 = \{x\}$, $T_2 = \{y\}$ and $T_3 = \{x, y\}$.

Q2.

$$(P)\quad \begin{cases} \max\ 0 \\ \forall j \in [\![1, m]\!],\ \sum_{i \in T_i} x_i = 1 \\ x_1, \dots, x_n \geq 0 \end{cases} \qquad (Q)\quad \begin{cases} \min\ y \\ A^T y \geq 0. \\ y \geq 0 \end{cases}$$

$$A = \left(\mathbb{1}_{i \in T_j} \right)$$