TD n° 1.

Ex 1. Rocq

Ex 2. Rocq

Ex 3. If $\Gamma \vdash A$ is provable in NJ, take some proof tree in NJ for it, it is a valid proof tree in NJ + R, thus $\Gamma \vdash A$ is provable in NJ + R.

Conversely, take a proof tree π of $\Gamma \vdash A$ in NJ + R, by admissibility of R, every occurence of R in π can be replaced by part of a tree that doesn't use R. Doing this starting from the leaves yield a valid proof tree for $\Gamma \vdash A$ in NJ.

Ex 4.

1. Take a proof tree of $\Gamma, A, B \vdash C$. Replace each occurence of

$$\dfrac{}{\Gamma, A, B, \Delta \vdash A}\,Ax \quad \text{with} \quad \dfrac{\dfrac{}{\Gamma, A\wedge B, \Delta \vdash A \wedge B}\,Ax}{\Gamma, A\wedge B, \Delta \vdash A}\,\wedge e_g$$

and

$$\dfrac{}{\Gamma, A, B, \Delta \vdash B}\,Ax \quad \text{with} \quad \dfrac{\dfrac{}{\tilde{\Delta} \vdash A \wedge B}\,Ax}{\tilde{\Delta} \vdash B}\,\wedge e_d,$$

also replacing every sequent $\Gamma, A, B, \Delta \vdash D$ with $\Gamma, A \wedge B, \Delta \vdash D$.

2. By induction on the size of Γ, we apply Q 1.

3. No, derivable rules cannot remove hypotheses (with the rules in NJ).

Ex 5.

1. $\dfrac{\Xi,\Gamma\vdash A}{\Xi,\Delta\vdash B}$ admissible iff $\left(\Xi,\Gamma\vdash A \text{ implies } \Xi,\Delta\vdash B\right)$

iff $\left(\Xi\vdash \bigwedge\Gamma\Rightarrow A \text{ implies } \Xi\vdash\bigwedge\Delta\Rightarrow B\right)$ Ex 4.

iff $\Xi\vdash(\bigwedge\Gamma\Rightarrow A)\Rightarrow(\bigwedge\Delta\Rightarrow B)$

iff $\Xi\vdash(\bigwedge\Gamma\Rightarrow A\wedge\bigwedge\Delta\Rightarrow B$ currying

2. Similarly to Q1, we know

$$\Xi,\Gamma\vdash(\bigwedge\Delta\Rightarrow A)\Rightarrow B$$

is provable, thus, by repeated currying, writing $\Delta=F_1,\dots,F_n$, the sequent

$$\Xi,\Gamma\vdash\underbrace{(F_1\Rightarrow\cdots\Rightarrow F_n\Rightarrow A)}_{\Phi}\Rightarrow B$$

is provable; take π a proof tree for it.

$$\dfrac{\dfrac{\pi}{\Xi,\Gamma\vdash\Phi\Rightarrow B}\qquad \dfrac{\dfrac{}{\Xi,\Gamma,\Delta\vdash A}\text{assumption}}{\Xi,\Gamma\vdash F_1\Rightarrow\cdots\Rightarrow F_n\Rightarrow A}\text{repeated }\Rightarrow_I}{\Xi,\Gamma\vdash B}\Rightarrow_E$$

Ex 6.

1.

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma\vdash A}}{\Gamma,\Delta\vdash A}\text{wk}\qquad\dfrac{\dfrac{\pi_2}{\Delta\vdash B}}{\Gamma,\Delta\vdash B}\text{wk}}{\Gamma,\Delta\vdash A\wedge B}\wedge I$$

2.

$$\dfrac{\dfrac{\Gamma\vdash\neg A}{\Gamma,\Delta\vdash\neg A}\text{wk}\qquad\dfrac{\Delta\vdash A}{\Gamma,\Delta\vdash A}\text{wk}}{\Gamma,\Delta\vdash\bot}\neg E$$

$$\dfrac{\dfrac{\Gamma\vdash A\vee B}{\Gamma,\Delta,\Sigma\vdash A\vee B}\text{wk}\qquad\dfrac{\Delta,A\vdash C}{\Gamma,\Delta,\Sigma,A\vdash C}\text{wk}\qquad\dfrac{\Sigma,B\vdash C}{\Gamma,\Delta,\Sigma,B\vdash C}\text{wk}}{\Gamma,\Delta,\Sigma\vdash C}\vee E$$

$$\dfrac{\dfrac{\Gamma \vdash A \Rightarrow B}{\Gamma, \Sigma \vdash A \Rightarrow B}\,\text{wR} \qquad \dfrac{\Delta \vdash B}{\Gamma, \Delta \vdash B}\,\text{wR}}{\Gamma, \Sigma \vdash B}\Rightarrow E.$$

Ex 7.

Define $C_\top := \bot \Rightarrow \bot$ and $C_{\neg A} := A \Rightarrow \bot$.

$$\dfrac{\dfrac{}{\Gamma, \bot \vdash \bot}\,\text{Ax}}{\Gamma \vdash C_\top} \Rightarrow I \quad \text{"}\top I\text{"}$$

$$\dfrac{\Gamma \vdash C_{\neg A} \qquad \Gamma \vdash A}{\Gamma \vdash \bot} \Rightarrow E$$

$\neg E.$

$$\dfrac{\Gamma, A \vdash \bot}{\Gamma \vdash C_{\neg A}} \Rightarrow I \quad \neg I$$

Ex 8. Rocq

Ex 9. Rocq

$\neg_R A$ is equivalent to $\neg A$ iff $\vdash \neg R$ is provable.