Assignment # 5

Hugo SALOU

Q1. The quantum phase estimation circuit happens in 4 phases.

- phase 1 : initialization $|0^m\rangle \otimes |\varphi\rangle$
- phase 2 : apply Hadamard gates $H^{\otimes m} \otimes \mathbb{1}$

$$\left(\frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} |k\rangle\right) \otimes |\varphi\rangle$$

- phase 3 : apply the controlled U^k $(\mathbb{1} \otimes I_m)$

as $U^k |\varphi\rangle = \left(e^{2i\pi\theta}\right)^k |\varphi\rangle$

$$\frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} e^{2i\pi k\theta} |k\rangle \otimes |\varphi\rangle$$

- phase 4 : apply the inverse Quantum Fourier Transform $QFT^{-1} \otimes \mathbb{1}$ (to the m first qubits)

$$\frac{1}{\sqrt{2^m}} \sum_{j=0}^{2^m-1} \frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} e^{2i\pi k\theta} \omega^{-kj} |j\rangle \otimes |\varphi\rangle$$

where $\omega = e^{2i\pi/2^m}$

After simplifications, we get, on the first m qubits,

$$|\psi\rangle := \frac{1}{2^m} \sum_{j=0}^{2^m-1} \sum_{k=0}^{2^m-1} e^{2i\pi k(\theta - j/2^m)} |j\rangle.$$

Q2. $$p_j = |\langle\psi|j\rangle|^2 = \frac{1}{2^{2m}} \left| \sum_{k=0}^{2^m-1} \left(e^{2i\pi(\theta - j/2^m)}\right)^k \right|^2$$

by geometric sum

$$= \frac{1}{2^{2m}} \left| \frac{1 - e^{2i\pi(2^m\theta - j)}}{1 - \underbrace{e^{2i\pi(\theta - j/2^m)}}_{\neq 1 \text{ as } \theta \neq j/2^m \ \forall j \in [\![0, 2^m-1]\!]}} \right|^2$$

Q3. We have that, if $|\theta - j/2^m| \geq 2^{-t-1}$, then

$$p_j \leq \frac{1}{2^{2m}} \frac{2^2}{(4 \times (\theta - j/2^m))^2} = 1/4 \cdot 2^{2m} (\theta - j/2^m)^2 \leq 2^{2(t-m)}$$

$\forall \gamma \in [-\pi, \pi]$, $\frac{2}{\pi} |\gamma| \leq |1 - e^{i\gamma}| \leq 2$

So the probability of measuring an "invalid value" $|\theta - j/2^m| \geq 2^{t-1}$ for any value is $\leq 2^m \times 2^{2t-2m} = 2^{2t-m}$.

Thus, the success probability is $\geq 1 - 2^{2t-m} \xrightarrow[t \text{ fixed}]{m \to \infty} 1.$

End of Assignment #5.