— *Homework* —

Semantics and Verification

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1 Toward Stone Duality.

Question 1. *Show that every Stone space (X, Ω) is Hausdorff (if $x, y \in X$ are distinct, there there are disjoint $U, V \in \Omega$ such that $x \in U$ and $y \in V$).*

Let $x, y \in X$ be two distinct points of a Stone space (X, Ω). As, (X, Ω) is T_0 and without loss of generality, there exists $W \in \Omega$ such that $x \in W$ and $y \notin W$. As (X, Ω) is zero-dimensional, we can write $W =: \bigcup_{i \in I} W_i$ where $W_i \in \mathbf{K}\Omega$ for every $i \in I$. Thus, there exists a clopen set $U := W_i \in \Omega$ such that $x \in W_i \subseteq U$. Define $V := X \setminus U \in \Omega$, and we have that $x \in U$, $y \in V$ (as $y \notin W \supseteq U$) and the open sets U and V are disjoint. We can conclude that every Stone space is Hausdorff.

Question 2. *Show that $\leq$ is a partial order on $\mathfrak{L}(\mathsf{LML})$.*

We start by showing the following lemma.

Lemma 1. We have $\phi \leq \psi$ if and only if $[\![\phi]\!] \subseteq [\![\psi]\!]$.

Proof. We have that $\phi \leq \psi$ iff $\phi \equiv \phi \wedge \psi$ iff $[\![\phi]\!] = [\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$ (that last equality is by definition of $[\![-]\!]$) iff $[\![\phi]\!] \subseteq [\![\psi]\!]$. □

We can thus easily show that $\leq$ is a partial order.

- ▷ *Reflexivity*. As $[\![\phi]\!] \subseteq [\![\phi]\!]$, we have that $\phi \leq \phi$ for every $\phi \in \mathfrak{L}(\mathsf{LML})$.
- ▷ *Transitivity*. For any $\phi, \psi, \vartheta \in \mathfrak{L}(\mathsf{LML})$, if $\phi \leq \psi$ and $\psi \leq \vartheta$ then, by the lemma, $[\![\phi]\!] \subseteq [\![\psi]\!] \subseteq [\![\vartheta]\!]$, thus we have $[\![\phi]\!] \subseteq [\![\vartheta]\!]$, *i.e.* $\phi \leq \vartheta$.
- ▷ *Antisymmetry*. For any $\phi, \psi \in \mathfrak{L}(\mathsf{LML})$, if $\phi \leq \psi$ and $\psi \leq \phi$ then, by double inclusion with the above lemma, $[\![\phi]\!] = [\![\psi]\!]$ thus $\phi = \psi$ as we consider LML-formulae quotiented by $\equiv$.

2 Lattices and Boolean Algebras.

2.1 Semilattices.

Question 3. *Let $(L, \leq)$ be a partial order.*

1. *Show that $(L, \leq)$ is a meet semilattice if, and only if, L has binary meets $\wedge : L \times L \to L$ and greatest element $\top \in L$.*

2. *Show that $(L, \leq)$ is a join semilattice if, and only if, L has binary joins $\vee : L \times L \to L$ and least element $\bot \in L$.*

1. If $(L, \leq)$ is a meet semilattice, then L has binary meets and a greatest element $\top = \bigwedge \emptyset$ (any element is a lower bound of $\emptyset$, thus the greatest lower bound of $\emptyset$ is the greatest element).

Now, suppose $(L, \leq)$ has a binary meet $\wedge$ and a greatest element $\top$. Consider $\{a_i \mid i \in I\}$ a finite subset of elements of L. By induction on $\#I \in \mathbb{N}$, we define $\bigwedge_{i \in I} a_i \in I$ and show that $\bigwedge_{i \in I} a_i$ is a meet of the finite set $\{a_i \mid i \in I\}$ (like the notation suggests).

- ▷ Define $\bigwedge_{i \in \emptyset} a_i := \top \in L$; as any element is a lower bound of $\emptyset$, the greatest lower bound of $\emptyset$ is the greatest element.

- ▷ Consider $I := J \sqcup \{i\}$. By induction hypothesis, we have that $\bigwedge_{j \in J} a_j$ exists in L and is a meet of $\{a_j \mid j \in J\}$ in $(L, \leq)$. Define

$$\bigwedge_{k \in I} a_k := \Big(\bigwedge_{j \in J} a_j\Big) \wedge a_i \in I.$$

  We have that $\bigwedge_{k \in I} a_k$ is a lower bound of $\{a_k \mid k \in I\}$. Consider an element a_k with $k \in I$. If $k \in J$ then $a_k \leq \bigwedge_{j \in J} a_j \leq \bigwedge_{k' \in I} a_{k'}$. Otherwise $k = i$ and we immediacy have that $a_i \leq \bigwedge_{k' \in I} a_{k'}$.

  Consider a lower bound $b \in L$ of $\{a_k \mid k \in I\}$, then b is a lower bound of $\{a_j \mid j \in J\}$ and $b \leq a_i$. We have $b \leq \bigwedge_{j \in J} a_j$ and $b \leq a_i$, therefore $b \leq \bigwedge_{k \in I} a_k$.

  We can conclude that $\bigwedge_{k \in I} a_k$ is a meet of $\{a_k \mid k \in K\}$.

Finally, we have that $(L, \leq)$ has finite meets.

2. This results follows from 1 when considering the partial order $(L, \geq)$, by duality. Meets in $(L, \geq)$ are exactly joins in $(L, \leq)$, and the greatest element of $(L, \geq)$ is the least element of $(L, \leq)$, and *vice versa*.

> **Note.** In the following, when I will be dealing with multiple partial orders on the same set (*e.g.* $\leq$ and $\geq$), I will write $\bigwedge_{\leq}$ for the meet operator in poset $(I, \leq)$, $\bigvee_{\leq}$ for the join operator in poset $(I, \leq)$, $\top_{\leq}$ for the greatest element in poset $(I, \leq)$ and $\bot_{\leq}$ for the least element in poset $(I, \leq)$.

Question 4. *Prove the following.*

1. *Let $(L, \leq)$ be a meet semilattice with binary meets $\wedge : L \times L \to L$ and greatest element $\top \in L$. Then $(L, \wedge, \top)$ is a commutative mooned in which every element is idempotent. Moreover, we have $a \leq b$ iff $a = a \wedge b$.*

2. *Let $(L, \leq)$ be a join semilattice with binary joins $\vee : L \times L \to L$ and least element $\bot$. Then $(L, \vee, \bot)$ is a commutative mooned in which every element is idempotent. Moreover, we have $a \leq b$ iff $b = a \wedge b$.*

1. Let $a, b, c \in L$. First, we have that $a \wedge b = \bigwedge\{a, b\} = \bigwedge\{b, a\} = b \wedge a$ thus the binary meet operation $\wedge$ is commutative. Then, as a special case of the previous question, we have that a and $\top \wedge a$ are both meets of $\{a\}$. And, by unicity of meets (*i.e.* antisymmetry of $\leq$, mainly), they are equals. Also as a special case of the previous question, we have that elements
$$a \wedge (b \wedge c) = \top \wedge (a \wedge (b \wedge c))$$
and
$$(a \wedge b) \wedge c = c \wedge (a \wedge b) = \top \wedge (c \wedge (a \wedge b))$$
are both meets of the set $\{a, b, c\}$, thus are equal. Next, we have that
$$a \wedge a = \bigwedge\{a, a\} = \bigwedge\{a\} = \top \wedge a = a$$
(penultimate equality is from last question), thus $a \wedge a = a$. Finally, we have that:
   - ▷ if $a = a \wedge b$ then a is a lower bound of $\{a, b\}$ thus $a \leq b$;

- ▷ if $a \leq b$ then $a = a \wedge b$ as a is a lower bound of $\{a, b\}$ and any lower bound c of $\{a, b\}$ must satisfy $c \leq a$.

2. Consider the meet semilattice $(L, \geq)$ and apply the results above. Meets in $(L, \geq)$ are exactly joins in $(L, \leq)$, and the greatest element of $(L, \geq)$ is the least element of $(L, \leq)$, and *vice versa*. The last statement follows from the equivalence:

$$a \leq b \quad \text{iff} \quad b \geq a \quad \text{iff} \quad b = a \wedge_{\geq} b \quad \text{iff} \quad b = a \vee_{\leq} b,$$

where the second "iff" follows from the result above for $(L, \geq)$, and the last one follows from the equality $a \wedge_{\geq} b = a \vee_{\leq} b$.

Question 5. *Prove the following.*

1. *Given a commutative monoid $(L, \wedge, \top)$ in which every element is idempotent, let $a \leq_{\wedge} b$ iff $a = a \wedge b$. Then $(L, \leq_{\wedge})$ is a meet semilattice with binary meets given by $\wedge$ and greatest element $\top$.*
2. *Given a commutative monoid $(L, \vee, \bot)$ in which every element is idempotent, let $a \leq_{\vee} b$ iff $b = a \vee b$. Then $(L, \leq_{\vee})$ is a join semilattice with binary joins given by $\vee$ and least element $\bot$.*

1. Let us start by showing that $(L, \leq_{\wedge})$ is a partial order.

- ▷ *Reflexivity.* As $a \wedge a = a$ by idempotence, we have $a \leq_{\wedge} a$.
- ▷ *Antisymmetry.* If $a \leq_{\wedge} b$ and $b \leq_{\wedge} a$ then, by commutativity, we have $a \wedge b = a = b$.
- ▷ *Transitivity.* If $a \leq_{\wedge} b$ and $b \leq_{\wedge} c$ then, by associativity,

$$a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c,$$

thus $a \leq_{\wedge} c$.

By question 3, it suffices to show that $(L, \leq_{\wedge})$ that binary meets for poset $(L, \leq_{\wedge})$ are $\wedge$ and that $\top$ is the greatest element of poset $(L, \leq_{\wedge})$. Consider a, b, c three arbitrary elements of L.

- ▷ For any $b \in L$, we have $b \wedge \top = b$ (as $\top$ is a neutral element) and thus $b \leq_{\wedge} \top$ for all $b \in L$, so $\top$ is the greatest element of $(L, \leq_{\wedge})$.

▷ Firstly, element $a \wedge b$ is a lower bound of $\{a, b\}$ as

$$\begin{aligned} a \wedge b \leq_\wedge a \quad &\text{iff} \quad a \wedge b = (a \wedge b) \wedge a \\ a \wedge b \leq_\wedge b \quad &\text{iff} \quad a \wedge b = (a \wedge b) \wedge b \end{aligned}$$

and the latter equalities are true by idempotence, associativity, and finally commutativity. Secondly, consider $c \in L$ such that we have $c \leq_\wedge a$ and $c \leq_\wedge b$, then $c \wedge a = c = c \wedge b$. We therefore have that $c \leq_\wedge a \wedge b$, as

$$c \wedge (a \wedge b) = (c \wedge a) \wedge b = c \wedge b = c.$$

We can conclude that $\wedge$ is the binary meet operator in $(L, \leq_\wedge)$.

2. Applying the previous result with the commutative monoid $(L, \vee, \bot)$, we obtain that $(L, \geq_\vee)$[a] is a meet semilattice where binary meets for $\geq_\vee$ are given by $\vee$ and the greatest element for $\geq_\vee$ is $\bot$. We can thus conclude that $(L, \leq_\vee)$ is a join semilattice where binary joins for $\leq_\vee$ are given by $\vee$ and the least element for $\leq_\vee$ is $\bot$.

Question 6. *Show the following, for the partial order $(\mathfrak{L}(\mathsf{LML}), \leq)$:*

1. $(\mathfrak{L}(\mathsf{LML}), \leq)$ *is a meet semilattice with greatest element $\top$ and binary joins given by*

$$\begin{aligned} - \wedge - : \mathfrak{L}(\mathsf{LML}) \times \mathfrak{L}(\mathsf{LML}) &\longrightarrow \mathfrak{L}(\mathsf{LML}) \\ (\phi, \psi) &\longmapsto \phi \wedge \psi\,; \end{aligned}$$

2. $(\mathfrak{L}(\mathsf{LML}), \leq)$ *is a join semilattice with least element $\bot$ and binary joins given by*

$$\begin{aligned} - \vee - : \mathfrak{L}(\mathsf{LML}) \times \mathfrak{L}(\mathsf{LML}) &\longrightarrow \mathfrak{L}(\mathsf{LML}) \\ (\phi, \psi) &\longmapsto \phi \vee \psi. \end{aligned}$$

We will use the lemma proven in question 2 (lemma 1, page 1).

[a] The notation is, in a way, "context-sensitive," as for an arbitrary monoid $(M, \circledast, \mathbf{I})$, we can either define $a \leq_\circledast b$ as $a \circledast b = a$ or $a \leq_\circledast b$ as $a \circledast b = b$.

1. We only need to show that $- \wedge -$ defines a binary meet for $(\mathfrak{L}(\mathsf{LML}), \leq)$ and that $\top$ is a greatest element.

   For any $\phi \in \mathfrak{L}(\mathsf{LML})$, we have $\phi \leq \top$ as $[\![\phi]\!] \subseteq [\![\top]\!] = (\mathbf{2}^{\mathrm{AP}})^{\omega}$, thus $\top$ is the greatest element.

   For any formulae $\phi, \psi \in \mathfrak{L}(\mathsf{LML})$, we have that $\phi \wedge \psi \leq \phi$ and $\phi \wedge \psi \leq \psi$ as both $[\![\phi]\!]$ and $[\![\psi]\!]$ are supersets of $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$ (by definition of interpretation $[\![-]\!]$). Then, if $\vartheta \leq \phi$ and $\vartheta \leq \psi$, we have that $[\![\vartheta]\!] \subseteq [\![\phi]\!]$ and $[\![\vartheta]\!] \subseteq [\![\psi]\!]$ thus $[\![\vartheta]\!] \subseteq [\![\phi]\!] \cap [\![\psi]\!] = [\![\phi \wedge \psi]\!]$, therefore $\vartheta \leq \phi \wedge \psi$.

   We can conclude that $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a meet semilattice with greatest element $\top$ and binary meets given by $- \wedge -$.

2. We only need to show that $- \vee -$ defines a binary join for $(\mathfrak{L}(\mathsf{LML}), \leq)$ and that $\bot$ is a least element.

   For any $\phi \in \mathfrak{L}(\mathsf{LML})$, we have $\bot \leq \phi$ as $\emptyset \subseteq [\![\bot]\!] \subseteq [\![\phi]\!]$, thus $\bot$ is the least element.

   For any formulae $\phi, \psi \in \mathfrak{L}(\mathsf{LML})$, we have that $\phi \leq \phi \vee \psi$ and $\psi \leq \phi \vee \psi$ as both $[\![\phi]\!]$ and $[\![\psi]\!]$ are subsets of $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$ (by definition of interpretation $[\![-]\!]$). Then, if $\phi \leq \vartheta$ and $\psi \leq \vartheta$, we have that $[\![\phi]\!] \subseteq [\![\vartheta]\!]$ and $[\![\psi]\!] \subseteq [\![\vartheta]\!]$ thus $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!] \subseteq [\![\vartheta]\!]$, therefore $\phi \vee \psi \leq \vartheta$.

   We can conclude that $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a join semilattice with least element $\bot$ and binary joins given by $- \vee -$.

Question 7. *Show that a map of meet* (resp. *join*) *semilattices is monotone.*

Let $f : L \to L'$ be an arbitrary function where $(L, \leq)$ and $(L', \leq')$ are partial orders.

1. Suppose $f : (L, \leq) \to (L', \leq')$ is a map of meet semilattices. Let $a, b \in L$. If $a \leq b$, then $a \wedge b = a$ and, as f preserves finite meets,

$$f(a) \wedge' f(b) = f(a \wedge b) = f(a),$$

   and thus $f(a) \leq' f(b)$. Therefore, f is monotone.

2. Suppose $f : (L, \leq) \to (L', \leq')$ is a map of join semilattices. Let $a, b \in L$. If $a \leq b$, then $a \vee b = b$ and, as f preserves finite joins,

$$f(a) \vee' f(b) = f(a \vee b) = f(b),$$

and thus $f(a) \leq' f(b)$. Therefore, f is monotone.

2.2 Lattices.

Question 8. *Consider the partial order $(L, \sqsubseteq)$ where*

$$L := \mathbb{N} \cup \{\alpha, \beta, \top\},$$

where $\sqsubseteq$ is the reflexive-transitive closure of $\sqsubset$, where

$$a \sqsubset b \quad \textit{iff} \quad \begin{cases} a < b \textit{ in } \mathbb{N} \\ \qquad \textit{or} \\ a \in \mathbb{N} \textit{ and } b \in \{\alpha, \beta\} \\ \qquad \textit{or} \\ a \in \{\alpha, \beta\} \textit{ and } b = \top. \end{cases}$$

Show that $(L, \sqsubseteq)$ is a join semilattice but is not a lattice.

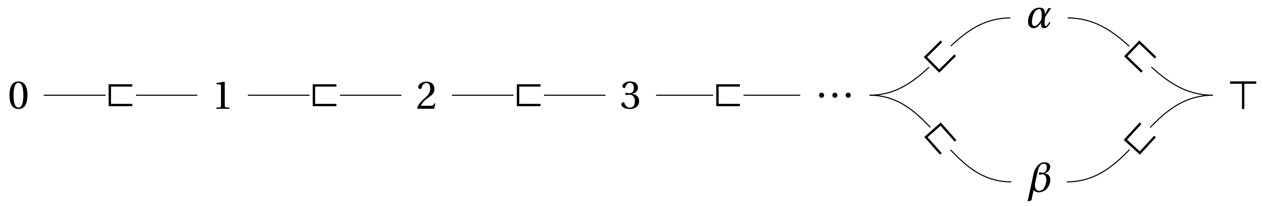


Figure 1 | *Hasse diagram of $(L, \sqsubseteq)$ from question 8*

Note: Hasse diagrams are usually read bottom-to-top, but this one is drawn left-to-right for convenience.

The relation $\sqsubseteq$ is a partial order. Reflexivity and transitivity is true by definition of $\sqsubseteq$ as the reflexive and transitive closure of $\sqsubseteq$. For antisymmetry, we have that:

- for $n, m \in \mathbb{N}$, $n \sqsubseteq m$ iff $n \leq m$;

- ▷ for any $n \in \mathbb{N}$ and $m \in L \setminus \mathbb{N}$, we have $n \sqsubseteq m$ and $m \not\sqsubseteq n$;
- ▷ $\alpha \sqsubseteq \top, \beta \sqsubseteq \top, \top \not\sqsubseteq \alpha, \top \not\sqsubseteq \beta, \alpha \not\sqsubseteq \beta$ and $\beta \not\sqsubseteq \alpha$;

(this can be shown by induction on the relation $\sqsubseteq$).

We have that 0 is the least element in $(L, \sqsubseteq)$: we have that $0 \sqsubseteq a$ for all $a \in L$. For $a, b \in L$, we can define $a \vee b$ as:

- ▷ if $a, b \in \mathbb{N}$, let $a \vee b := \min_{\leq_{\mathbb{N}}}(a, b)$;
- ▷ if $a \in \mathbb{N}$ and $b \in L \setminus \mathbb{N}$, let $a \vee b, b \vee a := b$;
- ▷ otherwise let $\alpha \vee \beta := \top$, $a \wedge a := a$, $a \wedge \top, \top \wedge a := \top$ for $a \in \{\alpha, \beta, \top\}$.

Using the previous results on $\sqsubseteq$, we have that $- \vee -$ *really* is a join.

This concludes the proof that $(L, \sqsubseteq)$ is a join semilattice.

We also have that $(L, \sqsubseteq)$ is not a lattice. Suppose it is a lattice, and consider the element $a := \alpha \wedge \beta$. Necessarily, we have that $a \in \mathbb{N}$ (if $a = \alpha$ then we would have $\alpha \sqsubseteq \beta$, which is false). As $a = \alpha \wedge \beta$ and $a + 1 \sqsubseteq \alpha, \beta$ we have, by definition of meet, that $a + 1 \sqsubseteq a$, thus $a + 1 \leq a$ (since $a, a + 1 \in \mathbb{N}$) which is ***absurd***. We can conclude that $(L, \sqsubseteq)$ is not a lattice.

Question 9. *Consider a set L equipped with two binary operations $\wedge, \vee : L \times L \to L$ and two constants $\top, \bot \in L$. Assume that $(L, \wedge, \top)$ and $(L, \vee, \bot)$ are commutative monoids in which every element is idempotent. Show that the following are equivalent.*

1. *The partial order $\leq_\vee$ induced by $(L, \vee, \bot)$ coincides with the partial order $\leq_\wedge$ induces by $(L, \wedge, \top)$.*
2. *$(L, \vee, \wedge, \bot, \top)$ satisfies the two following **absorptive laws**:*

$$\forall a, b \in L, \quad a \vee (a \wedge b) = a \tag{abs$_1$}$$
$$\forall a, b \in L, \quad a \wedge (a \vee b) = a \tag{abs$_2$}$$

- ▷ Let us show that 1 implies 2. Let $a, b \in L$. We have that $a \wedge b \leq_\wedge a$ and, assuming $\leq_\wedge$ and $\leq_\vee$ coincide, $a \wedge b \leq_\vee a$, thus $a \vee (a \wedge b) = a$, *i.e.* (abs$_2$) holds. Similarly, $a \leq_\vee a \vee b$ thus $a \leq_\wedge a \vee b$, so $(a \wedge b) \vee a = a$ holds, and we can recover (abs$_1$) by using commutativity.
- ▷ Let us show that 2 implies 1.

- Suppose $b \leq_\wedge a$, then $b \wedge a = b$. By (abs₁) and commutativity, we have $b \vee a = (b \wedge a) \vee a = a$, thus $b \leq_\vee a$.
- Suppose $b \leq_\vee a$, then $b \vee a = a$. By (abs₂), we have

  $$b \wedge a = b \wedge (b \vee a) = b,$$

  thus $b \leq_\wedge a$.

Thus the two order coincide.

Question 10. *Show that the partial order $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a lattice.*

We have shown that $(\mathfrak{L}(\mathsf{LML}), \leq)$ has a greatest element $\top$, a least element $\bot$, binary meets given by $- \wedge -$ and binary joins given by $- \vee -$ (question 6). Thus it has all finite meets and finite joins (as seen in question 3), *i.e.* $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a lattice.

Question 11. *Show that the function*

$$\begin{aligned} \bigcirc : \mathfrak{L}(\mathsf{LML}) &\longrightarrow \mathfrak{L}(\mathsf{LML}) \\ \phi &\longmapsto \bigcirc\phi \end{aligned}$$

is a morphism of lattices.

We know $\bigcirc$ is a map of meet iff $\bigcirc\top = \top$ and $\bigcirc(\phi \wedge \psi) = \bigcirc\phi \wedge \bigcirc\psi$. Both are true as,

$$\begin{aligned} [\![\bigcirc\top]\!] &= \left\{\sigma \in (\mathbf{2}^{\mathrm{AP}})^\omega \mid \sigma \upharpoonright 1 \in [\![\top]\!] = (\mathbf{2}^{\mathrm{AP}})^\omega\right\} = (\mathbf{2}^{\mathrm{AP}})^\omega = [\![\top]\!] \\ [\![\bigcirc(\phi \wedge \psi)]\!] &= \left\{\sigma \in (\mathbf{2}^{\mathrm{AP}})^\omega \mid \sigma \upharpoonright 1 \in [\![\phi]\!] \cap [\![\psi]\!]\right\} = [\![\bigcirc\phi]\!] \cap [\![\bigcirc\psi]\!] = [\![\bigcirc\phi \wedge \bigcirc\psi]\!]. \end{aligned}$$

Very similarly, $\bigcirc$ is a map of joins iff $\bigcirc\bot = \bot$ and $\bigcirc(\phi \vee \psi) = \bigcirc\phi \vee \bigcirc\psi$. One can show that both equalities hold by applying $[\![-]\!]$ and showing the equality of the sets like above.

Thus $\bigcirc : (\mathfrak{L}(\mathsf{LML}), \leq) \to (\mathfrak{L}(\mathsf{LML}), \leq)$ is a morphism of lattices.

2.3 Distributive Lattices.

Question 12. *Show that the following two* ***distributive laws*** *are equivalent in a lattice* $(L, \vee, \wedge, \bot, \top)$*:*

$$\forall a, b, c \in L, \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{dist$_1$}$$
$$\forall a, b, c \in L, \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{dist$_2$}$$

Suppose (dist$_1$) holds and let us show (dist$_2$) is true for $a, b, c \in L$:

$$\begin{aligned}
(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) && \text{by (dist}_1\text{)}\\
&= a \vee ((a \vee b) \wedge c) && \text{by (abs}_2\text{)}\\
&= a \vee (a \wedge b) \vee (b \wedge c) && \text{by (dist}_1\text{)}\\
&= a \vee (b \wedge c) && \text{by (abs}_1\text{).}
\end{aligned}$$

To prove that (dist$_1$) holds when (dist$_2$) is true, we can apply the previous result to the lattice $(L, \leq)^{\mathrm{op}} = (L, \geq)$. This gives exactly the implication "(dist$_2$) implies (dist$_1$)," as wanted.

Thus, the two distributive laws (dist$_1$) and (dist$_2$) are equivalent.

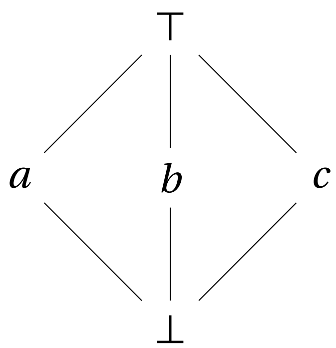
Question 13. *Show that the lattice* $(\mathfrak{L}(\mathsf{LML}), \leq)$ *is distributive.*

Let $\phi, \psi, \vartheta \in \mathfrak{L}(\mathsf{LML})$. We have that

$$[\![\phi \wedge (\psi \vee \vartheta)]\!] = [\![\phi]\!] \cap ([\![\psi]\!] \cup [\![\vartheta]\!]) = ([\![\phi]\!] \cap [\![\psi]\!]) \cup ([\![\phi]\!] \cap [\![\vartheta]\!]) = [\![(\phi \wedge \psi) \vee (\phi \wedge \vartheta)]\!],$$

thus $\phi \wedge (\psi \vee \vartheta) = (\phi \wedge \psi) \vee (\phi \wedge \vartheta)$.

Question 14. *Consider the following lattice* M_3*:*



(i.e. $\bot \le a, b, c \le \top$ with a, b, c incomparable). Show that M_3 is not distributive.

Suppose M_3 is distributive. As a, b, c are incomparable, we have that

$$a \wedge b = a \wedge c = \bot \qquad \text{and} \qquad b \vee c = \top,$$

and thus,

$$a = a \wedge \top = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = \bot \vee \bot = \bot,$$

which is ***absurd***. Thus M_3 is not distributive.

2.4 Booleans algebras.

Question 15. *Show that if $(L, \le)$ is a distributive lattice then $a \in L$ has at most one complement.*

Consider $c, c' \in L$ two complements of $a \in L$. Then, we have that

$$c = c \wedge \top = c \wedge (a \vee c') \overset{(\text{dist}_1)}{=} (c \wedge a) \vee (c \wedge c') = \bot \vee (c \wedge c') = c \wedge c',$$

and,

$$c' = c' \wedge \top = c' \wedge (a \vee c) \overset{(\text{dist}_1)}{=} (c' \wedge a) \vee (c' \wedge c) = \bot \vee (c' \wedge c) = c' \wedge c.$$

We can conclude that $c = c'$ by commutativity of meets.

Question 16. *Show that $(\mathfrak{L}(\mathsf{LML}), \le)$ is a Boolean algebra.*

Let us show that $\neg\phi$ is a complement for $\phi \in \mathfrak{L}(\mathsf{LML})$. We have to check that $\phi \wedge \neg\phi = \bot$ and $\phi \vee \neg\phi = \top$ hold. Both equalities can be easily checked with interpretations:

$$[\![\phi \wedge \neg\phi]\!] = [\![\phi]\!] \cap [\![\phi]\!]^\complement = \varnothing = [\![\bot]\!],$$

and

$$[\![\phi \vee \neg\phi]\!] = [\![\phi]\!] \cup [\![\phi]\!]^\complement = (\mathbf{2}^{\mathrm{AP}})^\omega = [\![\top]\!].$$

Thus, $\neg\phi$ is ***the*** complement of ϕ in $(\mathfrak{L}(\mathsf{LML}), \le)$, which is, as a consequence, a Boolean algebra.

Question 17. *Show that the following* ***De Morgan Laws*** *hold in every Boolean algebra* $(B, \vee, \wedge, \bot, \top)$*:*

$$a \wedge b = \neg(\neg a \vee \neg b) \qquad a \vee b = \neg(\neg a \wedge \neg b) \qquad a = \neg\neg a.$$

We start by showing that $\neg a \vee \neg b$ is a complement to $(a \wedge b)$:

$$(a \wedge b) \wedge (\neg a \vee \neg b) = (a \wedge b \wedge \neg b) \vee (a \wedge b \wedge \neg a) = \bot \vee \bot = \bot$$
$$(a \wedge b) \vee (\neg a \vee \neg b) = (a \vee \neg a \vee \neg b) \wedge (b \vee \neg a \vee \neg b) = \top \wedge \top = \top,$$

by using the Boolean algebra laws. Thus $a \wedge b = \neg(\neg a \vee \neg b)$.

For $a \vee b = \neg(\neg a \wedge \neg b)$, we proceed by duality by applying the previous result to the Boolean algebra $(B, \wedge, \vee, \top, \bot)$, as a complement for a in $(B, \geq)$ is exactly a complement for a in $(B, \leq)$.

We can easily check that $\neg\top = \bot$, and then

$$a = a \wedge \top = \neg(\neg a \vee \neg\top) = \neg(\neg a \vee \bot) = \neg\neg a.$$

Question 18. *Show that if f is a map of Boolean algebras from $(B, \leq)$ to $(B', \leq')$ then f preserves complements.*

We have that

$$\bot' = f(\bot) = f(a \wedge \neg a) = f(a) \wedge' f(\neg a),$$

and

$$\top = f(\top) = f(a \vee \neg a) = f(a) \vee' f(\neg a),$$

thus $f(\neg a)$ is a complement of $f(a)$ and, by unicity, $f(\neg a) = \neg f(a)$. We can conclude that a map of Boolean algebras preserves complements.

3 Representation of Boolean Algebras.

The rest will be given in the next part.